Introduction Robotics

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WTB Dynamics and Control
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New Plan for the Lecture Rooms

- Every Tuesday, $3^e+4^e$ hours
- Week 37+40+42+43 in AUD.10
- Week 38+39 in AUD.14
- Week 41 in AUD.15
Outline

• Recapitulation
• Parameterization of rotations
• Rigid motions
• Homogenous transformations
• Robotic manipulators
• Forward kinematics problem
Recapitulation
Representing rotations in 3-D

Each axis of the frame $o_1x_1y_1z_1$ is projected onto $o_0x_0y_0z_0$:

$$R^0_1 = \begin{bmatrix}
  x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\
  x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\
  x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 
\end{bmatrix}$$

$$R^0_1 \in SO(3)$$
Basic rotation matrices

\[
R_{x,\theta} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

\[
R_{y,\theta} = \begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

\[
R_{z,\theta} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Composition of rotations w.r.t. the Current Frame

- Frame relative to which a rotation occurs is the current frame.

- Rotations with respect to the current frames:

  \[ o_0x_0y_0z_0 \xrightarrow{R_0^1} o_1x_1y_1z_1 \xrightarrow{R_1^2} o_2x_2y_2z_2 \]

- Composition:

  \[ R_2^0 = R_1^0R_2^1. \]
Composition of rotations w.r.t. the Fixed Frame

- We can select one fixed frame as the reference one.
- Let $o_0x_0y_0z_0$ be the fixed frame.
- Rotations with respect to $o_0x_0y_0z_0$:

  \[
  o_0x_0y_0z_0 \xrightarrow{\text{rotation r.t. } 0} o_1x_1y_1z_1 \xrightarrow{\text{rotation r.t. } 0} o_2x_2y_2z_2
  \]

- Rotation $R$ can be represented in the current frame $o_1x_1y_1z_1$:

  \[
  R_2^1 = (R_1^0)^{-1} RR_1^0. \quad \text{[see similarity transformations]}
  \]

- Composition:

  \[
  R_2^0 = R_1^0 R_2^1 = R_1^0 (R_1^0)^{-1} RR_1^0 = RR_1^0
  \]
Parameterization of rotations

Rigid motions

Homogenous transformations
Parameterization of rotations (1/4)

- A rigid body has at most 3 rotational degrees of freedom.
- Rotational transformation $R \in SO(3)$ has 9 elements $r_{ij}$ that are not mutually independent since

$$\sum_i r_{ij}^2 = 1 \quad j = 1, 2, 3 \quad \text{[columns are unit vectors]}$$

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0 \quad i \neq j \quad \text{[columns are unit mutually orthogonal]}$$

- Given constraints define 6 independent equations with 9 unknowns; consequently, there are only 3 free variables.
Parameterization of rotations (2/4)

Euler angles

\[ R_{ZYZ} = R_{z,\phi}R_{y,\theta}R_{z,\psi} = \begin{bmatrix}
    c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\
    s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\
    -s_\theta c_\psi & s_\theta s_\psi & c_\theta
\end{bmatrix} \]

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Parameterization of rotations (3/4)

Roll, pitch, yaw angles

$XYZ$—yaw-pitch-roll angle transformation:

$$R = R_{z,\phi}R_{y,\theta}R_{x,\psi} = \begin{bmatrix} c_{\phi}c_{\theta} & -s_{\phi}c_{\psi} + c_{\phi}s_{\theta}s_{\psi} & s_{\phi}s_{\psi} + c_{\phi}s_{\theta}c_{\psi} \\ s_{\phi}c_{\theta} & c_{\phi}c_{\psi} + s_{\phi}s_{\theta}s_{\psi} & -c_{\phi}s_{\psi} + s_{\phi}s_{\theta}c_{\psi} \\ -s_{\theta} & c_{\theta}s_{\psi} & c_{\theta}c_{\psi} \end{bmatrix}$$
Parameterization of rotations (4/4)

**Axis / angle representation**

\[ k = [k_x \ k_y \ k_z]^T \] (unit) axis vector in \( o_0x_0y_0z_0 \)

\[ R_{k,\theta} \text{ represents rotation of } \theta \text{ about } k \]

\[
R_{k,\theta} =
\begin{bmatrix}
  k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\
  k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\
  k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta
\end{bmatrix}
\]

where \( v_\theta = 1 - c_\theta \)
Rigid motions (1/2)

- Rigid motion: a pure translation + a pure rotation.
- A rigid motion is an ordered pair \((d, R)\) where \(d \in \mathbb{R}^3\) and \(R \in SO(3)\).
- The group of all rigid motions: special Euclidean group \(SE(3)\)
  \[
  SE(3) = \mathbb{R}^3 \times SO(3)
  \]
Rigid motions (2/2)

\[ p^1 = R_2^1 p^2 + d_{1,2}^1 \]

\[ p^0 = R_1^0 p^1 + d_{0,1}^0 \]

\[ p^0 = \underbrace{R_1^0 R_2^1 p^2}_{R_2^0} + \underbrace{R_1^0 d_{1,2}^1 + d_{0,1}^0}_{d_{0,2}^0} \]

\[ p^0 = R_2^0 p^2 + d_{0,2}^0 \]
Homogenous transformations (1/2)

- We have

\[ R_2^0 = R_1^0 R_2^1 \quad d_{0,2}^0 = R_1^0 d_{1,2}^1 + d_{0,1}^0. \]

- Note that

\[
\begin{bmatrix}
  R_1^0 & d_{0,1}^0 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  R_2^1 & d_{1,2}^1 \\
  0 & 1
\end{bmatrix} =
\begin{bmatrix}
  R_1^0 R_2^1 & R_1^0 d_{1,2}^1 + d_{0,1}^0 \\
  0 & 1
\end{bmatrix}.
\]

- Consequently, rigid motion \((d, R)\) can be described by matrix representing homogenous transformation:

\[ H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}. \]
Homogenous transformations (2/2)

- Since $R$ is orthogonal, we have

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}.$$ 

- We augment vectors $p^0$ and $p^1$ to get their homogenous representations

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \quad P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

and achieve matrix representation of coordinate transformation

$$P^0 = H_1^0 P^1.$$
Basic homogenous transformations

\[
\begin{align*}
\text{Trans}_{x,a} &= \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \text{Rot}_{x,\alpha} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\text{Trans}_{y,b} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \text{Rot}_{y,\beta} &= \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\text{Trans}_{z,c} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} & \text{Rot}_{z,\gamma} &= \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]
Robotic Manipulators
Robot manipulators

- Kinematic chain is series of links and joints.

- Types of joints:
  - rotary (revolute, $\theta$)
  - prismatic (translational, $d$).
Anthropomorphic geometry (RRR)

- Resembles geometry of human arm.
- Features at least 3 rotary joints.
- Also known as articulated kinematic scheme.

http://www.ifr.org/pictureGallery/robType.htm

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Spherical geometry (RRP)

- Robot axes form a polar coordinate system.

Stanford Arm - 1969
SCARA geometry (RRP)

- Robot having 2 parallel rotary joints to provide compliance in $x$-$y$ direction (suitable for assembly tasks); sufficiently rigid in $z$-direction.
- SCARA stands for Selective Compliant Assembly Robot Arm or Selective Compliant Articulated Robot Arm.
Cylindrical geometry (RPP)

- Robot axes form a cylindrical coordinate system.
Cartesian geometry (PPP)

- Robot having 3 prismatic joints, whose axes are coincident with a Cartesian coordinate frame.
Parallel geometry

- Robot whose arms have concurrent prismatic or revolute joints.
- Some subset of robot links form a closed kinematic chain.
- There are at least two open kinematic chains connecting the base to the end-effector.
Workspace per robot geometry (1/2)

- **Cartesian Robot**
- **Cylindrical Robot**
- **Spherical Robot**
Workspace per robot geometry (2/2)

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<th>Kinematic structure</th>
<th>Workspace</th>
<th>Photo</th>
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<td>SCARA Robot</td>
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<td>Articulated Robot</td>
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<tr>
<td>Parallel Robot</td>
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</table>
Forward Kinematics

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Scope

- Formulate problem of Forward Kinematics (FK) in robotics.
- Introduce Denavits-Hartenberg (DH) formalism to assign coordinate frames to robot joints.
- Solve FK problem using matrix manipulation of homogenous transformations.
Forward kinematics problem

- Determine position and orientation of the end-effector as function of displacements in robot joints.
Illustration: planar manipulator with 2 joints

• Position of end-effector:

\[
\begin{align*}
  x &= a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) \\
  y &= a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2) \\
  a_i &= \text{the length of } i\text{th link}
\end{align*}
\]

• Orientation of end-effector:

\[
R_2^0 = \begin{bmatrix}
  \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\
  \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2)
\end{bmatrix}
\]
Conventions (1/2)

1. there are $n$ joints and hence $n + 1$ links; joints $1, 2, \ldots, n$; links $0, 1, \ldots, n$,

2. joint $i$ connects link $i - 1$ to link $i$,

3. actuation of joint $i$ causes link $i$ to move,

4. link $0$ (the base) is fixed and does not move,

5. each joint has a single degree-of-freedom (dof):

$$q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{cases}$$

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6. frame $o_i x_i y_i z_i$ is attached to link $i$; regardless of motion of the robot, coordinates of each point on link $i$ are constant when expressed in frame $o_i x_i y_i z_i$

7. when joint $i$ is actuated, link $i$ and its attached frame $o_i x_i y_i z_i$ experience resulting motion.
DH convention for homogenous transformations (1/2)

• An arbitrary homogeneous transformation is based on 6 independent variables: 3 for rotation + 3 for translation.
• DH convention reduces 6 to 4, by specific choice of the coordinate frames.
• In DH convention, each homogeneous transformation has the form:

\[
A_i = \begin{bmatrix}
    c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\
    s_{\theta_i} & c_{\theta_i} & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & d_i \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & a_i \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & c_{\alpha_i} & -s_{\alpha_i} & 0 & 0 \\
    0 & s_{\alpha_i} & c_{\alpha_i} & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]
DH convention for homogenous transformations (2/2)

- Position and orientation of coordinate frame $i$ with respect to frame $i-1$ is specified by homogenous transformation matrix:

$$A_i = \begin{bmatrix}
  c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\
  s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\
  0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\
  0 & 0 & 0 & 1
\end{bmatrix}$$

where

- $a_i = \text{link length}$,
- $\alpha_i = \text{link twist}$,
- $d_i = \text{link offset}$,
- $\theta_i = \text{joint angle}$.  

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Mathematical rational (1/3)

- There exists unique homogenous transformation matrix $A_1$ that expresses coordinates from frame 1 relative to frame 0.

(DH1) $x_1 \perp z_0$

$\downarrow$

rotational part of $A$

(DH2) $x_1$ intersects $z_0$

$\downarrow$

translational part of $A$
Mathematical rational (2/3)

\( R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \); \( x_1 \perp z_0 \Rightarrow x_1 \cdot z_0 = 0 \)

- Each column and row of \( R_1^0 \) has unique length, implying
  \[
  (x_1 \cdot x_0)^2 + (x_1 \cdot y_0)^2 = 1 \quad \text{and} \quad (y_1 \cdot z_0)^2 + (z_1 \cdot z_0)^2 = 1
  \]
  and in turn there exist unique \( \theta \) and \( \alpha \) such that
  \[
  (x_1 \cdot x_0, x_1 \cdot y_0) = (\cos \theta, \sin \theta) \\
  (y_1 \cdot z_0, z_1 \cdot z_0) = (\sin \alpha, \cos \alpha)
  \]
- Other elements of \( R_1^0 \) ensure \( R_1^0 \in SO(3) \).
Mathematical rational (3/3)

(DH2) $x_1$ intersects $z_0$

- Displacement between the origins of frames 1 and 0 is linear combination of vectors $z_0$ and $x_1$:

$$o_1^0 = o_0^0 + dz_0^0 + ax_1^0 =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} + d
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} + a
\begin{bmatrix}
\cos \theta \\
\sin \theta \\
0
\end{bmatrix} =
\begin{bmatrix}
\cos \theta \\
\sin \theta \\
d
\end{bmatrix}.$$
Physical meaning of DH parameters

- Link length $a_i$ is distance from $z_{i-1}$ to $z_i$ measured along $x_i$.
- Link twist $\alpha_i$ is angle between $z_{i-1}$ and $z_i$ measured in plane normal to $x_i$ (right-hand rule).
- Link offset $d_i$ is distance from origin of frame $i-1$ to the intersection $x_i$ with $z_{i-1}$, measured along $z_{i-1}$.
- Joint angle $\theta_i$ is angle from $x_{i-1}$ to $x_i$ measured in plane normal to $z_{i-1}$ (right-hand rule).
DH convention to assign coordinate frames

1. Assign $z_i$ to be the axis of actuation for joint $i+1$ (unless otherwise stated $z_n$ coincides with $z_{n-1}$).

2. Choose $x_0$ and $y_0$ so that the base frame is right-handed.

3. Iterative procedure for choosing $o_i x_i y_i z_i$ depending on $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$ $(i=1, 2, \ldots, n-1)$:
   a) $z_{i-1}$ and $z_i$ are not coplanar; there is an unique shortest line segment from $z_{i-1}$ to $z_i$, perpendicular to both; this line segment defines $x_i$ and the point where the line intersects $z_i$ is the origin $o_i$; choose $y_i$ to form a right-handed frame,
   b) $z_{i-1}$ is parallel to $z_i$; there are infinitely many common normals; choose $x_i$ as the normal passes through $o_{i-1}$; choose $o_i$ as the point at which this normal intersects $z_i$; choose $y_i$ to form a right-handed frame,
   c) $z_{i-1}$ intersects $z_i$; axis $x_i$ is chosen normal to the plane formed by $z_i$ and $z_{i-1}$; it’s positive direction is arbitrary; the most natural choice of $o_i$ is the intersection of $z_i$ and $z_{i-1}$, however, any point along the $z_i$ suffices; choose $y_i$ to form a right-handed frame.

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Forward kinematics (1/2)

- Homogenous transformation matrix relating the frame \( o_i x_i y_i z_i \) to \( o_{i-1} x_{i-1} y_{i-1} z_{i-1} \):

\[
A_i = A_i(q_i)
\]

The matrix \( A_i \) specifies position and orientation of \( o_i x_i y_i z_i \) w.r.t. \( o_{i-1} x_{i-1} y_{i-1} z_{i-1} \).

- Homogenous transformation matrix \( T^i_j \) expresses position and orientation of \( o_j x_j y_j z_j \) with respect to \( o_i x_i y_i z_i \):

\[
T^i_j = A_{i+1} A_{i+2} ... A_{j-1} A_j
\]
Forward kinematics (2/2)

- Forward kinematics of a serial manipulator with $n$ joints can be represented by homogenous transformation matrix $H_n^0$ which defines position and orientation of the end-effector’s (tip) frame $o_n x_n y_n z_n$ relative to the base coordinate frame $o_0 x_0 y_0 z_0$:

$$H_n^0(q) = T_n^0(q) = A_1(q_1) \cdot \ldots \cdot A_n(q_n),$$

$$H_n^0(q) = \begin{bmatrix}
R_n^0(q) & x_n^0(q) \\
0_{3\times1} & 1
\end{bmatrix}$$

$$0_{3\times1} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix};$$

$$q = \begin{bmatrix}
q_1 \\
\cdots \\
q_n
\end{bmatrix}^T.$$
Case-study: RRR robot manipulator

\[ q_1 \]

\[ \alpha_1 \] - twist angle

\[ a_i \] - link lengths

\[ d_i \] - link offsets

\[ q_i \] - displacements

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**DH parameters of RRR robot manipulator**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$ [rad]</th>
<th>$a_i$ [m]</th>
<th>$q_i$</th>
<th>$d_i$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi/2$</td>
<td>0</td>
<td>$q_1$</td>
<td>$C_0C_1=0.56$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$P_1C_2=0.2$</td>
<td>$q_2$</td>
<td>$C_1P_1=0.169$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$P_2C_3=0.415$</td>
<td>$q_3$</td>
<td>$C_2P_2=0.09$</td>
</tr>
</tbody>
</table>

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Forward kinematics of RRR robot manipulator (1/2)

- Coordinate frame \( o_3x_3y_3z_3 \) is related with the base frame \( o_0x_0y_0z_0 \) via homogenous transformation matrix:

\[
T^0_3(q) = A_1(q)A_2(q)A_3(q) = \\
= \begin{bmatrix} R^0_3(q) & x^0_3(q) \\ 0_{1 \times 3} & 1 \end{bmatrix}
\]

where

\[
q = [q_1 \ q_2 \ q_3]^T \\
x^0_3(q) = [x \ y \ z]^T
\]

\( 0_{1 \times 3} = [0 \ 0 \ 0] \)
Forward kinematics of RRR robot manipulator (2/2)

• Position of end-effector:

\[
x = \cos q_1 [a_3 \cos (q_2 + q_3) + a_2 \cos q_2] + (d_2 + d_3) \sin q_1 \\
y = \sin q_1 [a_3 \cos (q_2 + q_3) + a_2 \cos q_2] - (d_2 + d_3) \cos q_1 \\
z = a_3 \sin (q_2 + q_3) + a_2 \sin q_2 + d_1
\]

• Orientation of end-effector:

\[
R_3^0 = \begin{bmatrix}
\cos q_1 \cos (q_2 + q_3) & -\cos q_1 \sin (q_2 + q_3) & \sin q_1 \\
\sin q_1 \cos (q_2 + q_3) & -\sin q_1 \sin (q_2 + q_3) & -\cos q_1 \\
\sin (q_2 + q_3) & \cos (q_2 + q_3) & 0
\end{bmatrix}
\]