analysis and synthesis of electronic circuits

linear circuit analysis

- the circuit simulation problem
  - problem formulation
  - general solution strategy
- the circuit equations
  - circuit theory
  - sparse tableau approach
  - nodal analysis
  - modified nodal analysis
- solving linear resistive circuits
  - gauss elimination
  - LU decomposition
    - pivoting for accuracy
  - sparse matrix techniques

the circuit analysis problem

- the circuit topology
  - the circuit graph with identification
- the branch relations
  - the parameters of the branch component
  - identification of the controlling quantitie(s)
- the state of the circuit
  - the energy stored in the circuit
- the stimuli to be applied
- the response demands
  - what quantities are required?
  - what numerical accuracy is required?

strategy for solving network equations

read the circuit and and compose the set of circuit equations

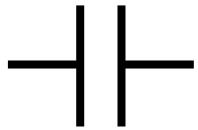
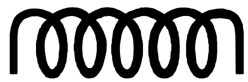
read the simulation commands

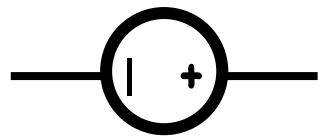
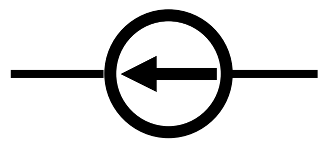
initialize the circuit solution

dynamic non-linear circuit:
 - solve the set of simultaneous differential equations

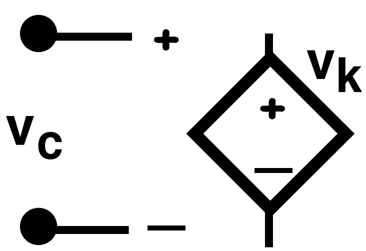
```
  determine the time discretization
  while (t < t_final)
  {    non-linear resistive circuit:
        - solve the set of simultaneaous algebraic equations
       estimate the solution
       do  {  linearize the equations around current solution
              linear resistive circuit:
               - solve the set of simultaneous linear equations
           }   (no convergence)
   print  (current solution)
  }
```

ideal elements

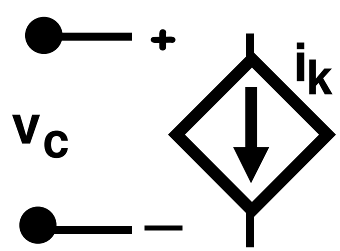
- resistor $v = R\,i$  or $i = i(v)$ or $v = v(i)$
- capacitor  $i = \left[C(v) + v\frac{dC}{dv}\right]\frac{dv}{dt}$
- inductor  $v = \left[L(i) + i\frac{dL}{di}\right]\frac{di}{dt}$

- voltage source  $v(t) = ES(t)$
- current source  $i(t) = IS(t)$

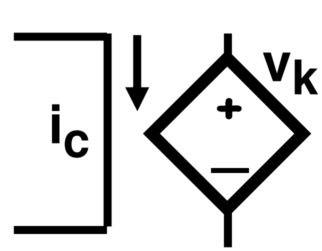
- coupled inductors
- controlled sources



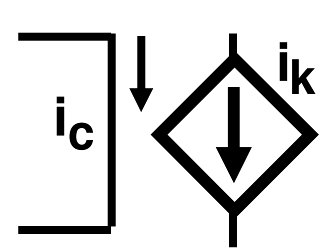
$v_k = E_k v_c$



$i_k = G_k v_c$

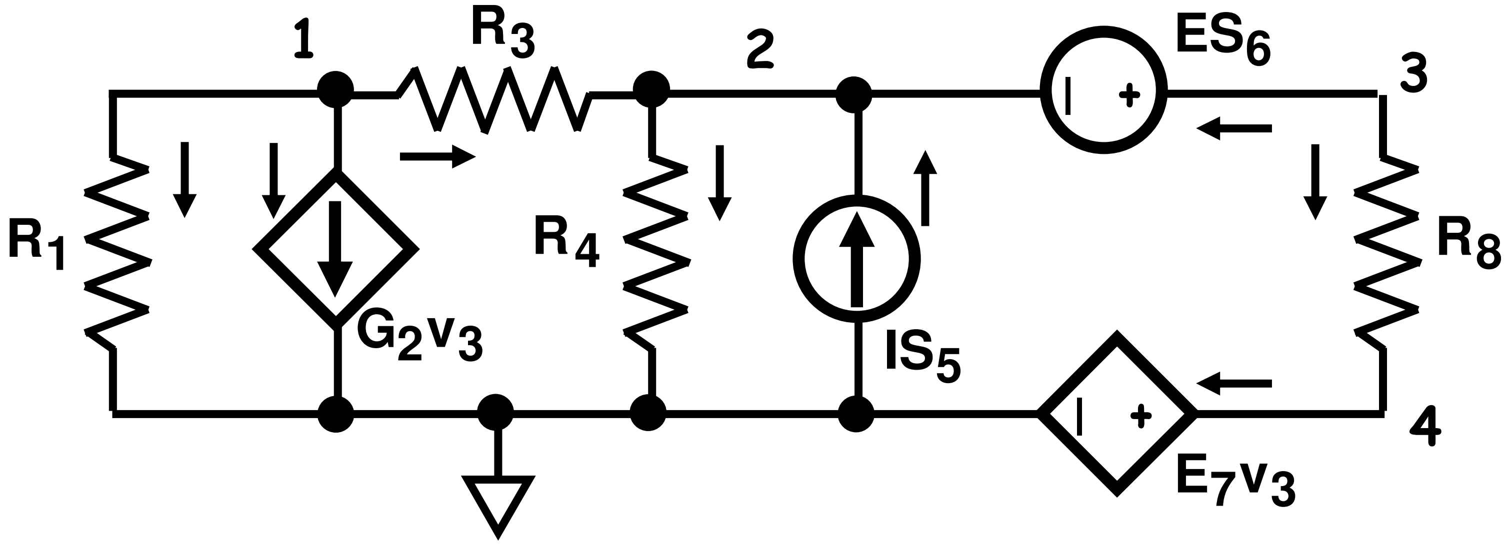


$v_k = H_k i_c$



$i_k = F_k i_c$

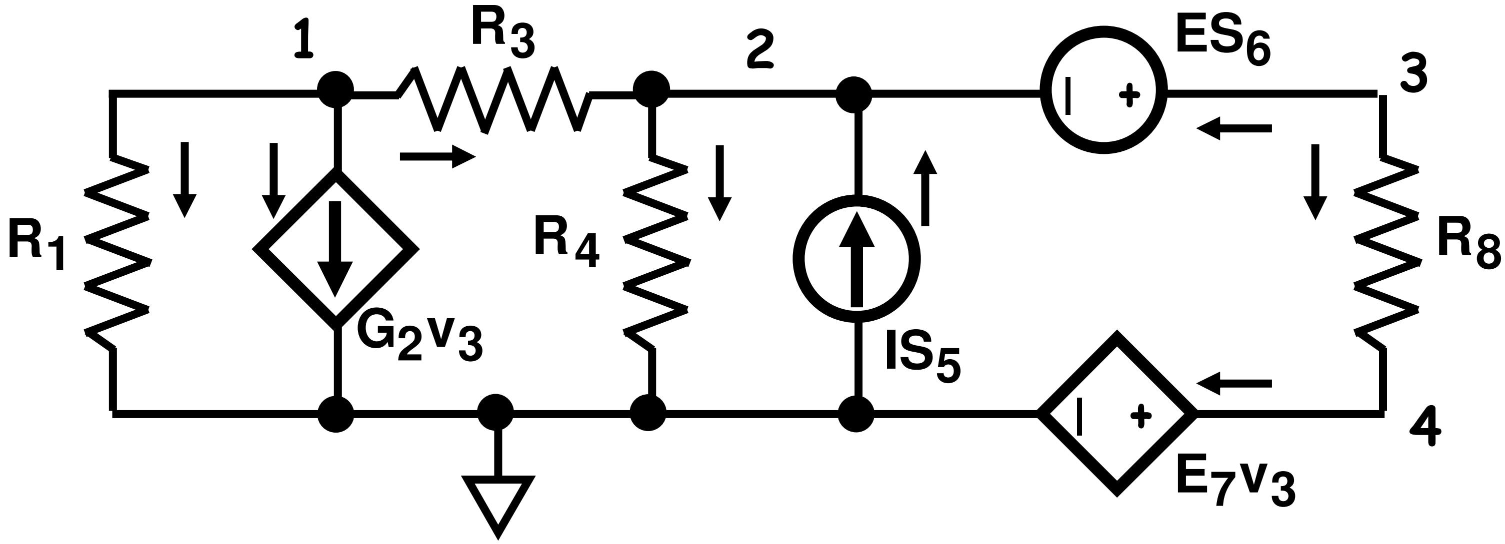
kirchhoff current laws



$A\ i = 0$

$$
\begin{array}{llllllllll}
i_1 & +i_2 & +i_3 & & & & & & = 0 \\
 & & -i_3 & +i_4 & -i_5 & -i_6 & & & = 0 \\
 & & & & & +i_6 & & +i_8 & = 0 \\
 & & & & & & +i_7 & -i_8 & = 0
\end{array}
$$

kirchhoff voltage laws



$$A\ i = 0$$

$$v - A^T e = 0$$

$$\begin{array}{lllllllllllll}
v_1 & & & & & & & & -e_1 & & & & = 0 \\
& v_2 & & & & & & & -e_1 & & & & = 0 \\
& & v_3 & & & & & & -e_1 & +e_2 & & & = 0 \\
& & & v_4 & & & & & & -e_2 & & & = 0 \\
& & & & v_5 & & & & & +e_2 & & & = 0 \\
& & & & & v_6 & & & & +e_2 & -e_3 & & = 0 \\
& & & & & & v_7 & & & & & -e_4 & = 0 \\
& & & & & & & v_8 & & & -e_3 & +e_4 & = 0
\end{array}$$

the complete sparse tableau

$$
\left(\begin{array}{cccccccc|cccccccc|cccc}
1 & 1 & 1 & & & & & & & & & & & & & & & & & \\
 & & -1 & 1 & -1 & -1 & & & & & & & & & & & & & & \\
 & & & & & 1 & & 1 & & & & & & & & & & & & \\
 & & & & & & 1 & -1 & & & & & & & & & & & & \\
\hline
 & & & & & & & & 1 & & & & & & & & -1 & & & \\
 & & & & & & & & & 1 & & & & & & & -1 & & & \\
 & & & & & & & & & & 1 & & & & & & -1 & 1 & & \\
 & & & & & & & & & & & 1 & & & & & & -1 & & \\
 & & & & & & & & & & & & 1 & & & & & 1 & & \\
 & & & & & & & & & & & & & 1 & & & & 1 & -1 & \\
 & & & & & & & & & & & & & & 1 & & & & & -1 \\
 & & & & & & & & & & & & & & & 1 & & & -1 & 1 \\
\hline
R_1 & & & & & & & & -1 & & & & & & & & & & & \\
 & 1 & & & & & & & & & -G_2 & & & & & & & & & \\
 & & R_3 & & & & & & & & -1 & & & & & & & & & \\
 & & & R_4 & & & & & & & & -1 & & & & & & & & \\
 & & & & 1 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & & -E_7 & & & & 1 & & & & & \\
 & & & & & & & R_8 & & & & & & & & -1 & & & &
\end{array}\right)
\left(\begin{array}{c}
i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ e_1 \\ e_2 \\ e_3 \\ e_4
\end{array}\right)
=
\left(\begin{array}{c}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ IS_5 \\ ES_6 \\ 0 \\ 0
\end{array}\right)
$$

the sparse tableau

$$\begin{pmatrix} A & O & O \\ O & I & -A^T \\ K_i & K_v & O \end{pmatrix} \begin{pmatrix} i \\ v \\ e \end{pmatrix} = \begin{pmatrix} o \\ o \\ s \end{pmatrix}$$

A is the reduced incidence matrix of the circuit graph!

- there 2b+n-1 equations for 2b+n-1 unknowns
  - b being the number of branches, n the number of nodes
- works for any linear resistive network
- it provides all node voltages and currents
- the tableau can be easily assembled by inspection
- the coefficient matrix is very sparse

however, sophisticated algorithms are needed to exploit sparsity and maintain sufficient accuracy, and only few responses are of interest!

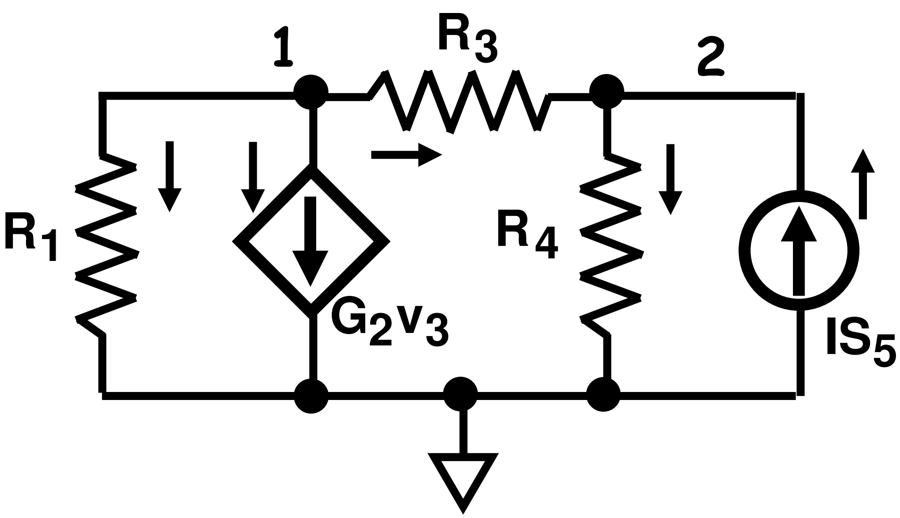
nodal analysis

kirchhoff current equations:

$$
\begin{aligned}
i_1 \quad + i_2 \quad + i_3 \quad &= 0 \\
- i_3 \quad + i_4 \quad - i_5 &= 0
\end{aligned}
$$

with branch relations:

$$
\begin{aligned}
\frac{1}{R_1} v_1 + G_2 v_3 + \frac{1}{R_3} v_3 &= 0 \\
-\frac{1}{R_3} v_3 + \frac{1}{R_4} v_4 - IS_5 &= 0
\end{aligned}
$$



and node potentials substituted:

$$
\begin{aligned}
\frac{1}{R_1} e_1 + G_2 (e_1 - e_2) + \frac{1}{R_3} (e_1 - e_2) &= 0 \\
-\frac{1}{R_3} (e_1 - e_2) + \frac{1}{R_4} e_2 &= IS_5
\end{aligned}
$$

or in matrix notation:

nodal admittance matrix

$$
Y_n \; e = \begin{pmatrix} \frac{1}{R_1} + G_2 + \frac{1}{R_3} & -G_2 - \frac{1}{R_3} \\ -\frac{1}{R_3} & \frac{1}{R_3} + \frac{1}{R_3} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ IS_5 \end{pmatrix} = IS
$$

$$
Y_n = -A K_i^{-1} K_v A^T
$$

from sparse tableau to nodal analysis

$$\begin{pmatrix} A & O & O \\ O & I & -A^T \\ I & -Y & O \end{pmatrix} \begin{pmatrix} i \\ v \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ IS \end{pmatrix}$$

$IS = K_i^{-1} s$

$Y = -K_i^{-1} K_v$

$$\begin{pmatrix} O & AY & O \\ O & I & -A^T \\ I & -Y & O \end{pmatrix} \begin{pmatrix} i \\ v \\ e \end{pmatrix} = \begin{pmatrix} -A\,IS \\ 0 \\ IS \end{pmatrix}$$

these are exactly the nodal analysis equations!

$$\begin{pmatrix} O & O & AYA^T \\ O & I & -A^T \\ I & -Y & O \end{pmatrix} \begin{pmatrix} i \\ v \\ e \end{pmatrix} = \begin{pmatrix} -A\,IS \\ O \\ IS \end{pmatrix}$$

- the sparse tableau approach cannot be less accurate or less efficient than nodal analysis
  - it is just a specific pivoting order!
- Y_n has non-zero diagonal, is even diagonally dominant
- Y_n is also very sparse, though relatively less than ST
- Y_n can be easily assembled by inspection

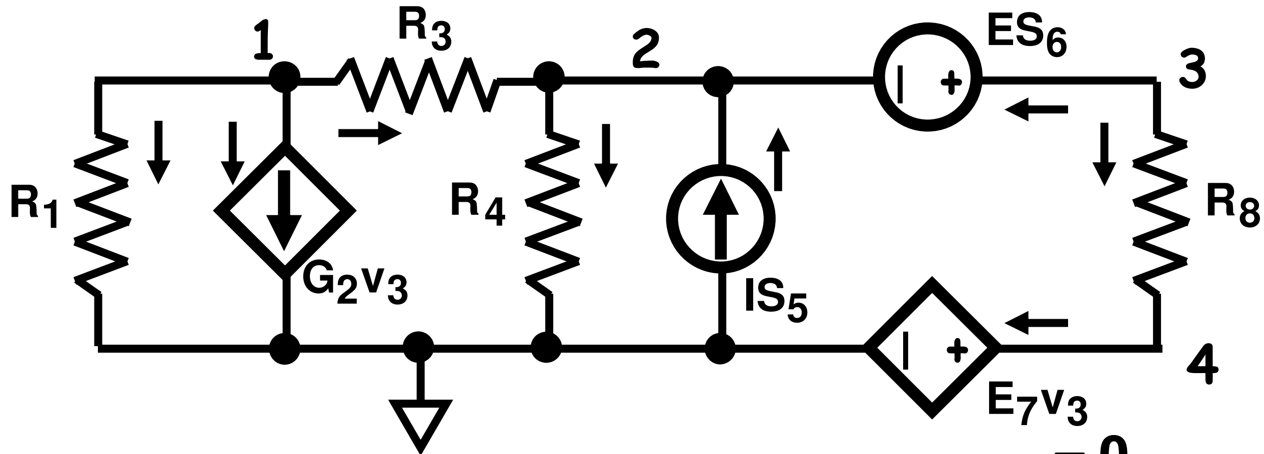
assembling by inspection

$$
\Delta Y_n = \begin{pmatrix}
 & & \cdot & & & \cdot & & \\
 & & \cdot & & & \cdot & & \\
\cdot & \cdot & \frac{1}{R_k} & \cdot & \cdot & -\frac{1}{R_k} & \cdot & \cdot \\
 & & \cdot & & & \cdot & & \\
 & & \cdot & & & \cdot & & \\
\cdot & \cdot & -\frac{1}{R_k} & \cdot & \cdot & \frac{1}{R_k} & \cdot & \cdot \\
 & & \cdot & & & \cdot & & \\
 & & \cdot & & & \cdot & &
\end{pmatrix}
\qquad
\Delta IS = \begin{pmatrix}
\cdot \\ \cdot \\ -IS_k \\ \cdot \\ \cdot \\ +IS_k \\ \cdot \\ \cdot
\end{pmatrix}
$$

$$
\Delta Y_n = \begin{pmatrix}
 & & & & \cdot & & & \cdot & & \\
 & & & & \cdot & & & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & G_k & \cdot & \cdot & -G_k & \cdot & \cdot \\
 & & & & \cdot & & & \cdot & & \\
 & & & & \cdot & & & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & -G_k & \cdot & \cdot & G_k & \cdot & \cdot \\
 & & & & \cdot & & & \cdot & & \\
 & & & & \cdot & & & \cdot & & \\
 & & & & \cdot & & & \cdot & & \\
 & & & & \cdot & & & \cdot & &
\end{pmatrix}
$$

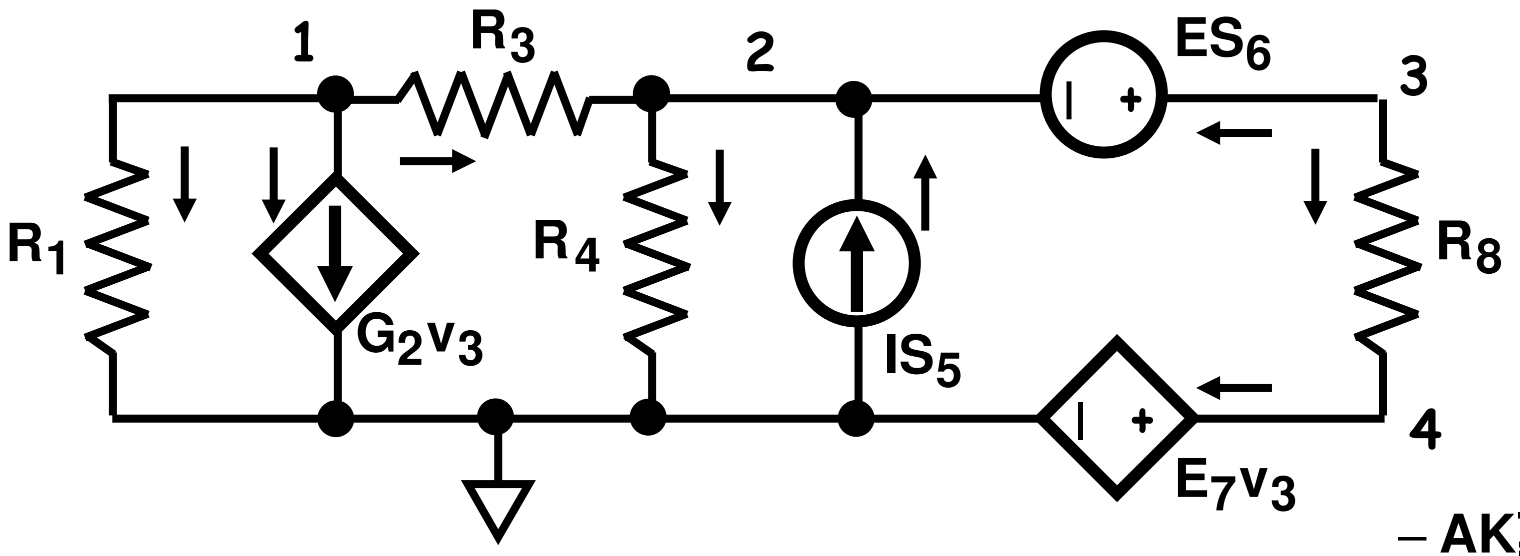
modified nodal analysis

- start with KCL
- use branch relations
- add missing relations



$$
\begin{array}{llllllllll}
i_1 & +i_2 & +i_3 & & & & & & & = 0 \\
 & & -i_3 & +i_4 & -i_5 & -i_6 & & & & = 0 \\
 & & & & & i_6 & & & +i_8 & = 0 \\
 & & & & & & & +i_7 & -i_8 & = 0 \\
\frac{1}{R_1} v_1 & +G_2 v_3 & +\frac{1}{R_3} v_3 & & & & & & & = 0 \\
 & & -\frac{1}{R_3} v_3 & +\frac{1}{R_4} v_4 & -IS_5 & -i_6 & & & & = 0 \\
 & & & & & i_6 & & & +\frac{1}{R_8} v_8 & = 0 \\
 & & & & & & & i_7 & -\frac{1}{R_8} v_8 & = 0 \\
 & & & & & & & & v_6 & = ES_6 \\
 & & & & & & & v_7 & -E_7 v_3 & = 0
\end{array}
$$

modified nodal analysis



$$-AK_i^{-1}K_vA^T = A\,IS$$

$$\begin{pmatrix} \frac{1}{R_1}+G_2+\frac{1}{R_3} & -G_2-\frac{1}{R_3} & 0 & 0 & 0 & 0 \\ -\frac{1}{R_3} & \frac{1}{R_3}+\frac{1}{R_4} & 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{R_8} & -\frac{1}{R_8} & 1 & 0 \\ 0 & 0 & -\frac{1}{R_8} & \frac{1}{R_8} & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -E_7 & E_7 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ i_6 \\ i_7 \end{pmatrix} = \begin{pmatrix} 0 \\ IS_5 \\ 0 \\ 0 \\ ES_6 \\ 0 \end{pmatrix}$$

solving linear resistive circuits

= solving a set of linear algebraic equations: $A\ x = b$

(A is a non-singular square matrix)

back substitution:

$$
\begin{aligned}
u_{11}x_1 \; + u_{12}x_2 \; + \dots\dots \; + u_{1n}x_n &= b_1 \\
u_{22}x_2 \; + \dots\dots \; + u_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad & \;\; \vdots \\
\vdots \qquad & \;\; \vdots \\
u_{nn}x_n &= b_n
\end{aligned}
$$

$$x_n = \frac{b_n}{u_{n,n}}$$

$$x_{n-1} = \frac{b_{n-1} - u_{n-1,n}x_n}{u_{n-1,n-1}}$$

. . . . .

. . . . .

. . . . .

$$x_1 = \frac{b_1 - u_{1,2}x_2 - \dots\dots - u_{1,n}x_n}{u_{1,1}}$$

$$x_j = \frac{b_j - \sum_{k=j+1}^{n} u_{j,k}x_k}{u_{j,j}}$$

forward substitution:

$$x_j = \frac{b_j - \sum_{k=1}^{j-1} l_{j,k}x_k}{l_{j,j}}$$

$\frac{1}{2}n(n+1)$ multiplications

gaussian elimination

$$\begin{array}{cccccc} a_{1,1}x_1 & +a_{1,2}x_2 & +\ldots\ldots & +a_{1,n}x_n & =b_1 \\ a_{2,1}x_1 & +a_{2,2}x_2 & +\ldots\ldots & +a_{2,n}x_n & =b_2 \\ : & : & : & : & : \\ : & : & : & : & : \\ a_{n,1}x_1 & +a_{n,2}x_2 & +\ldots\ldots & +a_{n,n}x_n & =b_n \end{array}$$

eliminate x_1, using

$$a_{i,j}^{(2)} = a_{i,j} - \frac{a_{i,1}}{a_{1,1}} a_{1,j}$$

$$b_i^{(2)} = b_i - \frac{a_{i,1}}{a_{1,1}} b_1$$

continue
with the smaller matrices
until an upper-triangular
set
is obtained

$$\begin{array}{cccccc} a_{1,1}x_1 & +a_{1,2}x_2 & +\ldots\ldots & +a_{1,n}x_n & =b_1 \\ 0 & +a_{2,2}^{(2)}x_2 & +\ldots\ldots & +a_{2,n}^{(2)}x_n & =b_2^{(2)} \\ : & : & : & : & : \\ : & : & : & : & : \\ 0 & +a_{n,2}^{(2)}x_2 & +\ldots\ldots & +a_{n,n}^{(2)}x_n & =b_n^{(2)} \end{array}$$

$$\begin{array}{cccccc} a_{1,1}^{(1)}x_1 & +a_{1,2}^{(1)}x_2 & +\ldots\ldots & +a_{1,n}^{(1)}x_n & =b_1^{(1)} \\ & +a_{2,2}^{(2)}x_2 & +\ldots\ldots & +a_{2,n}^{(2)}x_n & =b_2^{(2)} \\ & & : & : & : \\ & & & : & : \\ & & & +a_{n,n}^{(n)}x_n & =b_n^{(n)} \end{array}$$

n divisions

$$\frac{n^3+3n^2-n}{3}$$ multiplications

pivoting

- obviously all pivots have to be non-zero
  - if necessary, interchange rows and/or columns
  - this is always possible, since the tableaux are non-singular
  - more problems with STA than with MNA
- interchange needed for accuracy as well!

$$\begin{aligned} 0.000125x_1 + 1.25x_2 &= 6.25 \\ 12.5x_1 + 12.5x_2 &= 75 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 1.0001 \\ x_2 &= 4.9999 \end{aligned}$$

$$\begin{aligned} 0.000125x_1 + 1.25x_2 &= 6.25 \\ -1.25\times10^5 x_2 &= -6.25\times10^5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 5 \end{aligned}$$

$$\begin{aligned} +1.25x_2 &= 6.25 \\ 12.5x_1 + 12.5x_2 &= 75 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 1 \\ x_2 &= 5 \end{aligned}$$

- partial pivoting: row interchange only
- complete pivoting: row and column interchange

LU decomposition

$$\begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

if we set $U x = y$, then we can solve $A x = b$ by first solving $L y = b$ by forward substitution, and then, $U x = y$ by backward substitution.

IMPORTANT ADVANTAGE : LU decomposition independent of b

this means: for each right-hand side only n^2 multiplications suffice!

LU decomposition itself is solving $n^2 + n$ unknowns from n^2 equations:

$$\sum_{p=1}^{\min\{i,j\}} l_{ip} u_{pj} = a_{ij}$$

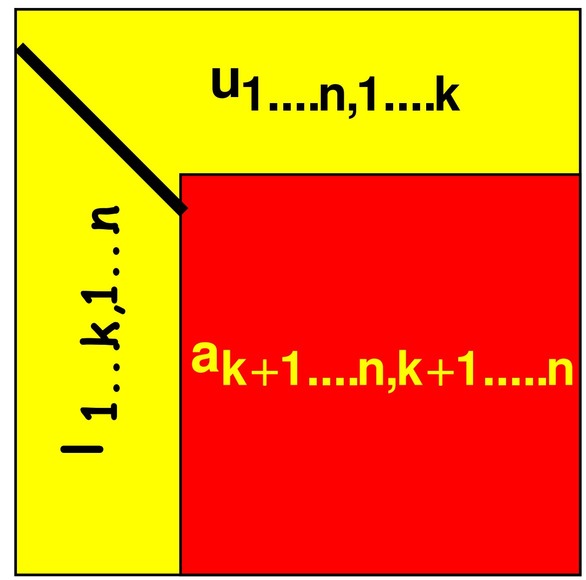
LU decomposition

set the diagonal elements of L equal to one, and proceed:

$$\begin{aligned} l_{11}u_{11} &= a_{11} \\ l_{11}u_{12} &= a_{12} \\ \vdots \quad & \quad \vdots \quad \vdots \\ l_{11}u_{1n} &= a_{1n} \end{aligned} \quad \Rightarrow \quad \begin{aligned} l_{21}u_{11} &= a_{21} \\ l_{31}u_{11} &= a_{31} \\ \vdots \quad & \quad \vdots \quad \vdots \\ l_{n1}u_{11} &= a_{n1} \end{aligned} \quad \Rightarrow \quad \begin{aligned} l_{22}u_{22} + l_{21}u_{12} &= a_{22} \\ l_{22}u_{23} + l_{21}u_{13} &= a_{23} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad & \quad \vdots \\ l_{22}u_{2n} + l_{21}u_{1n} &= a_{2n} \end{aligned} \quad \Rightarrow$$

in general:

$$u_{kj} = a_{kj} - \sum_{p=1}^{k-1} l_{kp}u_{pj} \qquad l_{ik} = \frac{a_{ik} - \sum_{p=1}^{k-1} l_{ip}u_{pj}}{u_{kk}}$$



ANOTHER ADVANTAGE:
LU decomposition can be done "in situ"!

it requires n divisions and $\frac{n^3 + n}{3}$ multiplications

in essence LU decomposition is just a way to do gaussian elimination

sparse matrix techniques

number of multiplications to solve Ax=b

$$\sum_{k=1}^{n-1}(r_k - 1)(c_k - 1) + \sum_{k=1}^{n-1}(c_k - 1) + \sum_{k=1}^{n-1}(r_k - 1) + n$$

r_k = # non-zeros in k-th row

c_k = # non-zeros in k-th row

since $a_{ij}^{(k+1)} = a_{ij}^{(k)} - \dfrac{a_{ik}^{(k)} a_{kj}^{(k)}}{a_{kk}^{(k)}}$

a fill-in is created in position (i,j) fif $a_{ik}^{(k)}$ **and** $a_{kj}^{(k)}$ **are non-zero**

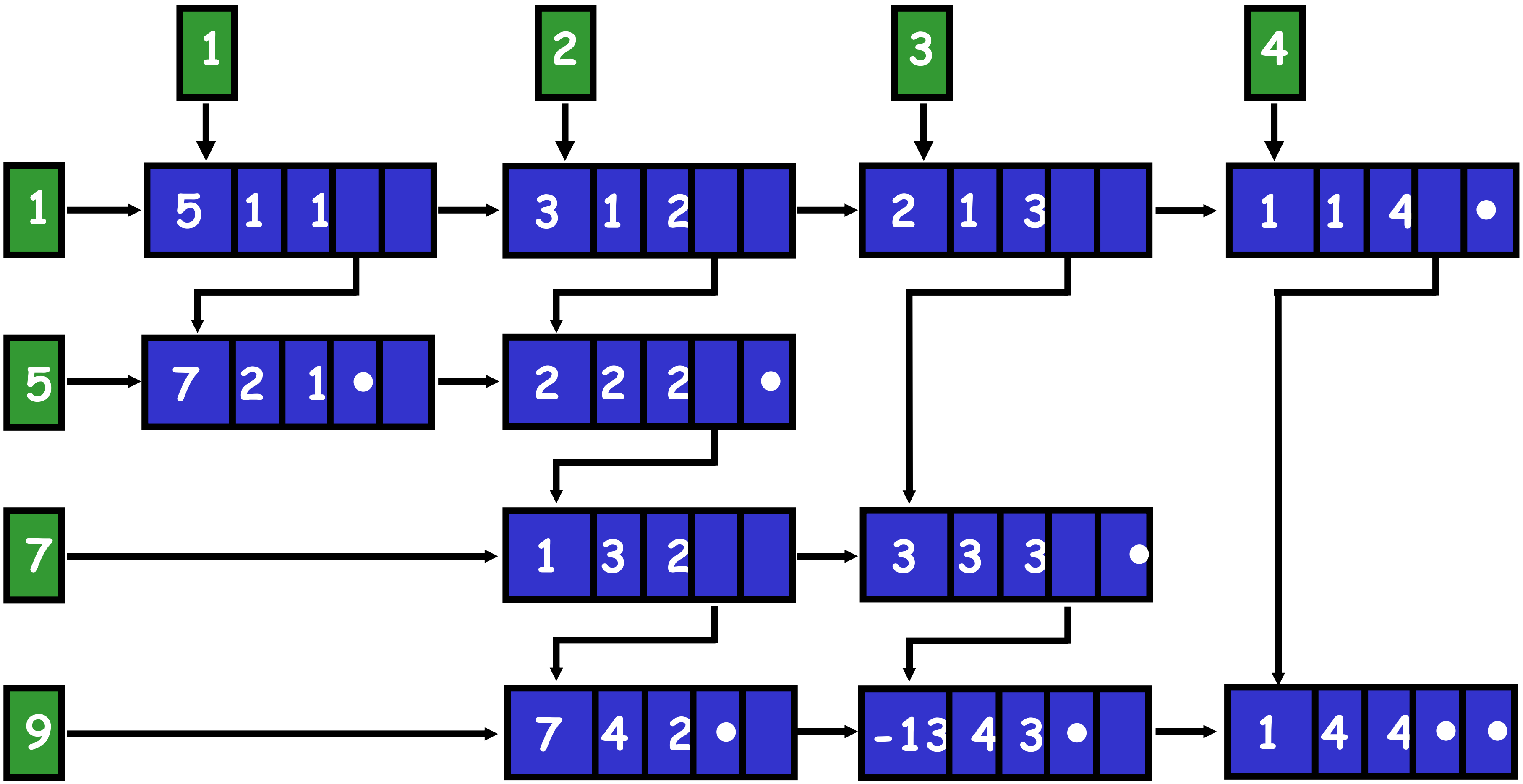
the minimum fill-in problem is NP-hard

markowitz-criterion :

- minimize $(r_i - 1)(c_j - 1)$
- in case of ties,
  choose the one with minimum row count

beware of numerical accuracy!!!

data structures



$$\begin{pmatrix} 5 & 3 & 2 & 1 \\ 7 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 7 & -13 & 1 \end{pmatrix}$$