nonlinear circuits

- solving non-linear resistive circuits
  - problem formulation
  - newton-raphson iteration
  - companion networks
  - non-convergence and overflow
  - complexity

the website: www.ics.ele.tue.nl/es/education/
strategy for solving network equations

read the circuit and and compose the set of circuit equations
read the simulation commands
initialize the circuit solution

dynamic non-linear circuit:
- solve the set of simultaneous differential equations
determine the time discretization

while (t < t_final)
{
  non-linear resistive circuit:
  - solve the set of simultaneous algebraic equations
  estimate the solution
  do
  {
    linearize the equations around current solution
    linear resistive circuit:
    - solve the set of simultaneous linear equations
  } (no convergence)
  print (current solution)
}
problem formulation

for sets of nonlinear algebraic equations
no necessary and sufficient conditions
for existence and/or uniqueness!

\[ F(x) = 0 \]

1. \( f_1(i,v,e) = i_1 + i_2 + i_3 = 0 \)
2. \( f_2(i,v,e) = -i_3 + i_4 - i_5 = 0 \)
3. \( f_3(i,v,e) = v_1 = 0 \)
4. \( f_4(i,v,e) = v_2 = 0 \)
5. \( f_5(i,v,e) = v_3 - e_1 = 0 \)
6. \( f_6(i,v,e) = v_4 - e_2 = 0 \)
7. \( f_7(i,v,e) = v_5 - e_2 = 0 \)
8. \( f_8(i,v,e) = v_1 - R_1 i_1 = 0 \)
9. \( f_9(i,v,e) = i_2 - l_2 \exp(v_2/V_T) - 1 = 0 \)
10. \( f_{10}(i,v,e) = v_3 - R_3 i_3 = 0 \)
11. \( f_{11}(i,v,e) = i_4 - l_4 \exp(v_4/V_T) - 1 = 0 \)
12. \( f_{12}(i,v,e) = i_5 = I_S \)
one-dimensional newton-raphson

\[ f(y) = f(y_0) + (y - y_0) \left. \frac{df}{dy} \right|_{y_0} + \text{higher order terms} \]

\[ y_1 = y_0 - f(y_0) \left[ \left. \frac{df}{dy} \right|_{y_0} \right]^{-1} \]

\[ y_2 = y_1 - f(y_1) \left[ \left. \frac{df}{dy} \right|_{y_1} \right]^{-1} \]

\[ y_{k+1} = y_k - f(y_k) \left[ \left. \frac{df}{dy} \right|_{y_k} \right]^{-1} \]

If \( f \) is twice continuously differentiable and \( \left. \frac{df}{dy} \right|_{y^*} \neq 0 \) then \( |y_k - y^*| \leq C|y_{k-1} - y^*|^2 \)

\[ z = f(y_0) + (y - y_0) \left. \frac{df}{dy} \right|_{y_0} \]

Newton-Raphson converges if the initial guess is close enough.
multi-dimensional newton-raphson

\( x^{k+1} = x^k - J^{-1}(x^k) f(x^k) \) or equivalently \( J(x^k)(x^{k+1} - x^k) = -f(x^k) \)

\[
J(x^k) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} |_{x^k} & \frac{\partial f_1}{\partial x_2} |_{x^k} & \cdots & \frac{\partial f_1}{\partial x_n} |_{x^k} \\
\frac{\partial f_2}{\partial x_1} |_{x^k} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} |_{x^k} & \frac{\partial f_n}{\partial x_2} |_{x^k} & \cdots & \frac{\partial f_n}{\partial x_n} |_{x^k}
\end{pmatrix}
\]

if \( J(x) \) is lipschitz continuous and non-singular, then

\[
\| x^k - x^* \| \leq C \| x^{k-1} - x^* \|^2
\]

provided \( x^0 \) is close enough to \( x^* \)
multi-dimensional newton-raphson

\[ x^{k+1} = x^k - J^{-1}(x^k) f(x^k) \]  

or equivalently  

\[ J(x^k)(x^{k+1} - x^k) = -f(x^k) \]

the newton-raphson iteration step:

1. evaluate \( f(x^k) \)
2. compute the jacobian matrix \( J(x^k) \)
3. determine the (linear set of equations)
4. solve for \( x^{k+1} - x^k \) by forward/backward substitution
5. determine \( x^{k+1} \)
6. check stopping criterion

if \( J(x) \) is lipschitz continuous and non-singular, then

\[ \|x^k - x^*\| \leq C \|x^{k-1} - x^*\|^2 \]

provided \( x^0 \) is close enough to \( x^* \)
the companion circuit

\[ \text{J}(x^k)(x^{k+1} - x^k) = -f(x^k) \]

\[ x_2 = i_2 \]
\[ x_7 = v_2 \]

\[
\begin{align*}
  f_1(i,v,e) &= i_1 + i_2 + i_3 = 0 \\
  f_2(i,v,e) &= -i_3 + i_4 - i_5 = 0 \\
  f_3(i,v,e) &= v_1 = 0 \\
  f_4(i,v,e) &= -e_1 = 0 \\
  f_5(i,v,e) &= v_2 = 0 \\
  f_6(i,v,e) &= -e_1 + e_2 = 0 \\
  f_7(i,v,e) &= v_3 = 0 \\
  f_8(i,v,e) &= -e_2 = 0 \\
  f_9(i,v,e) &= i_2 - i_2[\exp(v_2/V_T) - 1] = 0 \\
  f_{10}(i,v,e) &= v_3 - R_3i_3 = 0 \\
  f_{11}(i,v,e) &= i_4 - i_4[\exp(v_4/V_T) - 1] = 0 \\
  f_{12}(i,v,e) &= i_5 = IS_5
\end{align*}
\]
the companion circuit

\[ J(x^k)(x^{k+1} - x^k) = -f(x^k) \]

\[ x_2 = i_2 \]
\[ x_7 = v_2 \]

\[ \frac{\partial f_9}{\partial x_2}
    \bigg|_{x^k} = 1 \quad \text{and} \quad
\frac{\partial f_9}{\partial x_7}
    \bigg|_{x^k} = -\frac{I_2}{V_T} \exp\left(\frac{v_2^k}{V_T}\right) \]

\[ f_9(i, v, e) = i_2 - l_2 \left[ \exp\left(\frac{v_2}{V_T}\right) - 1 \right] = 0 \]
the companion circuit

\[
\frac{\partial f_g}{\partial x_2} \bigg|_{x^k} = 1 \quad \text{and} \quad \frac{\partial f_g}{\partial x_7} \bigg|_{x^k} = -\frac{I_2}{V_T} \exp\left(\frac{v_2^k}{V_T}\right)
\]

\[
1 \times (i_2^{k+1} - i_2^k) - \frac{I_2}{V_T} \exp\left(\frac{v_2^k}{V_T}\right) \times (v_2^{k+1} - v_2^k) = -i_2^k + I_2 \left[\exp\left(\frac{v_2^k}{V_T}\right) - 1\right]
\]

\[
i_2^{k+1} - \frac{I_2}{V_T} \exp\left(\frac{v_2^k}{V_T}\right) v_2^{k+1} = I_2 \left[\exp\left(\frac{v_2^k}{V_T}\right) - 1\right] - \frac{I_2}{V_T} \exp\left(\frac{v_2^k}{V_T}\right) v_2^k
\]

\[
i_2^{k+1} - G_2 v_2^{k+1} = IS_2(k)
\]
the companion circuit

\[ J(x^k)(x^{k+1} - x^k) = -f(x^k) \]

\[ i_2^{k+1} - G_v^{k+1} = IS_2(k) \]
newton-raphson in circuit simulators

- solving non-linear resistive circuits = solving a sequence of linear resistive circuits
  - linearization usual done with sparse tableau
  - once the companion matrix is obtained other methods apply
- solutions of subsequent circuits are assumed to be close
  - they have to be "close enough" for proper convergence
  - "time steps" should be "small enough"
- the circuit equations have to be assembled once
- the zero/non-zero pattern stays the same
  - pivoting for sparsity can be done once
- the values of the coefficient matrix change
  - pivoting for numerical accuracy may require changes
  - this may cause additional fill-ins
  - fill-in removal is usually not done
- the number of changes depends on the pivoting sequence
convergence

convergence occurs in the limit $\Rightarrow$ a stopping criterion

$$\left\| x^{k+1} - x^k \right\| < \epsilon_a + \epsilon_r \times \min\left\{ \left\| x^{k+1} \right\|, \left\| x^k \right\| \right\}$$

to be applied to currents and voltages of nonlinear elements

convergence problems due to

- discontinuous first derivatives
  - often a consequence of the model equations
    e.g. some mosfet models between triode and saturation
- the initial guess is not close enough to the solutions
  - often a consequence of switching events
  - remedies
    - time step control
    - optimization based modifications
    - continuation methods
overflow

mostly due the characteristics of active devices
e.g. the exponential characteristic of bipolar junctions

apply a limiting scheme
when the junction voltage
exceeds a critical value

\[ i^{k+1} = i(v_{lim}^{k+1}) \]

\[ \frac{I_s}{V_T} \exp\left(\frac{V_k}{V_T}\right) \times \left( v^{k+1} - v^k \right) + I_s \left[ \exp\left(\frac{V_k}{V_T}\right) - 1 \right] = I_s \left[ \exp\left(\frac{v_{lim}^{k+1}}{V_T}\right) - 1 \right] \]

example critical value:

\[ V_T \ln\left( \frac{V_T}{I_s \sqrt{2}} \right) \]
complexity reduction

- compute branch derivatives by finite differences
  - models are like small subnetworks
  - user can supply any (feasible model)
  - simple implementation
  - error prone and possibly numerically unstable
- provide analytical expressions for all derivatives
  - computing the jacobian becomes function evaluation
  - burden on the user
  - no guard against errors
- use a preprocessor to produce the derivatives
- built-in models with the derivative
  - faster evaluation
  - user cannot introduce models of his own
  - but limiting can be tailored to the model equations
- reduce the number of jacobian computations
  - broyden rank update of $J(x)$
  - do not update $J(x)$ each step (shamanskii algorithm)
  - bypassing when changes are small