# Sensing, Computing, Actuating
# Lecture 1 - Resistive displacement

**Exercise 1: Throttle position sensor**

A throttle position sensor (TPS) is a system that determines the ignition timing of an engine by comparing the position of the throttle valves with the rotational speed of the engine. The system is used in almost all modern gasoline engines. The throttle position sensor is normally located on the butterfly valve. The sensor is typically build using a potentiometer that provides a variable electrical resistance which is dependent on the position of the throttle valve. Figure 1 provides a schematic view of the throttle position sensor. The sensor consists of a variable resistor $R_T$ of which its resistance varies between 0 $\Omega$ (at $\Theta = 0°$) and $R_T$ $\Omega$ (at $\Theta = 270°$). The sensor is connected to a circuit that processes the repsonse from the sensor. This circuit has a purely resistive input-impedance $R_m = R_T/a$.

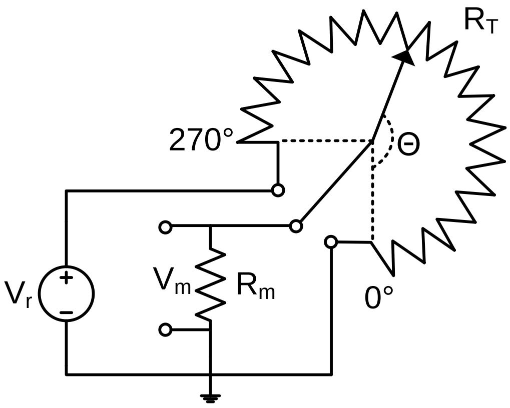


Figure 1: Throttle position sensor with processing circuit.

(a) Show that the voltage $V_m$ across the resistor $R_m$ in terms of the supply voltage $V_r$, the angle $\Theta$, the resistance $R_T$ and the constant $a$ is equal to:

$$V_m = \frac{270°\Theta}{(270°)^2 + a\Theta(270° - \Theta)} V_r$$

**Answer:** Assume that $\alpha = \frac{\Theta}{270°}$, then it holds that:

$$\frac{V_m}{V_r} = \frac{R_m//\alpha R_T}{R_m//\alpha R_T + (1-\alpha)R_T}$$

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{\frac{R_m \alpha R_T}{R_m + \alpha R_T}}{\frac{R_m \alpha R_T}{R_m + \alpha R_T} + (1-\alpha)R_T}$$

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{R_m \alpha R_T}{R_m \alpha R_T + (1-\alpha)R_T(R_m + \alpha R_T)}$$

Divide by $R_T$:

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{R_m \alpha}{R_m \alpha + (1-\alpha)(R_m + \alpha R_T)}$$

Substitute $R_m = R_T/a$:

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{(R_T/a)\alpha}{(R_T/a)\alpha + (1-\alpha)(R_T/a + \alpha R_T)}$$

Divide by $R_T$:

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{(1/a)\alpha}{(1/a)\alpha + (1-\alpha)(1/a + \alpha)}$$

Multiply with $a/a$:

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{\alpha}{\alpha + (1-\alpha)(1 + a\alpha)}$$

Substitute $\alpha = \frac{\Theta}{270^\circ}$:

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{\frac{\Theta}{270^\circ}}{\frac{\Theta}{270^\circ} + (1 - \frac{\Theta}{270^\circ})(1 + a\frac{\Theta}{270^\circ})}$$

Multiply with $270^\circ/270^\circ$:

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{\Theta}{\Theta + (270^\circ - \Theta)(1 + a\frac{\Theta}{270^\circ})}$$

Multiply again with $270^\circ/270^\circ$:

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{270^\circ\Theta}{270^\circ\Theta + (270^\circ - \Theta)(270^\circ + a\Theta)}$$

Rewrite numerator:

$$\Leftrightarrow \frac{V_m}{V_r} = \frac{270^\circ\Theta}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)}$$

The voltage across the resistor $R_m$ is thus equal to:

$$V_m = \frac{270^\circ\Theta}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)} V_r$$

(b) Show that the relative error $\epsilon$ in the output voltage $V_m$ of the sensor due to the loading resistance $R_m$ is equal to:

$$\epsilon = \frac{a\Theta(270^\circ - \Theta)}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)}$$

**Answer:** The relative error is defined as:

$$\epsilon = \left| \frac{V_m - V_o}{V_o} \right|$$

, with $v_m$ the voltage as compared in question (a) and $V_o$ the output voltage of the circuit with the loading resistor $R_m$. The open-circuit situation (circuit without $R_m$) is equivalent to the situation $a = R_T/R_m = 0$. Therefore it holds that:

$$V_o = \frac{\Theta}{270^\circ} V_r$$

The relative error is thus equal to:

$$\begin{aligned}
\epsilon &= \left|\frac{V_m}{V_o} - 1\right| \\
&= \left|\frac{270^\circ\Theta}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)}\frac{270^\circ}{\Theta} - 1\right| \\
&= \left|\frac{(270^\circ)^2}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)} - 1\right| \\
&= \left|\frac{(270^\circ)^2 - (270^\circ)^2 - a\Theta(270^\circ - \Theta)}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)}\right| \\
&= \left|\frac{-a\Theta(270^\circ - \Theta)}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)}\right|
\end{aligned}$$

It holds that $\Theta \leq 270^\circ$ en $a \geq 0$, the relative error is thus equal to:

$$\epsilon = \frac{a\Theta(270^\circ - \Theta)}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)}$$

(c) What ratio $R_T/R_m$ should the resistors $R_T$ and $R_m$ have such that the relative error $\epsilon$ in term of the output voltage $V_m$ due to the loading resistance $R_m$ is always smaller then 5%?

**Answer:** It must hold that:

$$\epsilon = \left|\frac{-a\Theta(270^\circ - \Theta)}{(270^\circ)^2 + a\Theta(270^\circ - \Theta)}\right| \leq 5\% = 0.05$$

Compute first for which value of $\Theta$ the error reaches its maximum. Compute for this purpose:

$$\frac{d\epsilon}{d\Theta} = 0$$

Use the quotient-rule:

$$\frac{d}{d\Theta}\left(\frac{u}{v}\right) = \frac{v\frac{du}{d\Theta} - u\frac{dv}{d\Theta}}{v^2}$$

$$\Rightarrow \frac{d\epsilon}{d\Theta} = \frac{\left((270^\circ)^2 + a\Theta(270^\circ - \Theta)\right)(a270^\circ - 2a\Theta) - (a\Theta(270^\circ - \Theta))(a270^\circ - 2a\Theta)}{\left((270^\circ)^2 + a\Theta(270^\circ - \Theta)\right)^2}$$

The maximal error is reached when:

$$\frac{d\epsilon}{d\Theta} = 0$$

This constraint is met when:

$$a270^\circ - 2a\Theta = 0$$

$$a = 0 \vee \Theta = 135^\circ$$

Since a finite resistance $R_m$ is used, it is impossible for $a$ to become equal to zero. The error reaches therefore its maximum at $\Theta = 135^\circ$. It must hold now that:

$$\epsilon = \frac{a135^\circ(270^\circ - 135^\circ)}{(270^\circ)^2 + a135^\circ(270^\circ - 135^\circ)} \leq 5\% = 0.05$$

$$\Leftrightarrow \frac{a(135°)^2}{(270°)^2 + a(135°)^2} \leq 0.05$$

$$\Leftrightarrow \frac{a}{4+a} \leq 0.05$$

$$\Leftrightarrow a \leq 0.05(4+a)$$

$$\Leftrightarrow 0.95a \leq 0.20$$

$$\Leftrightarrow a \leq 0.21$$

It holds that: $R_m = R_T/a$, and thus $a = R_T/R_m$.

$$\Rightarrow R_T/R_m \leq 0.21$$

(d) Assume that the potentiometer has a resistance of 250 kΩ. The potentiometer has been build by turning a wire with a length of 1 m around a tube that has an average diameter of 2 cm. The wire has a specific resistance of 20000 $\mu\Omega$cm. What is the resolution of the sensor in degrees?

**Answer:**
To determine the resolution, we must first determine the diameter of the wire. The diameter $d$ of the wire is equal to:

$$R = \rho\frac{l}{A} \Rightarrow 250k\Omega = 20000\mu\Omega cm\frac{1m}{\pi\,(d/2)^2}$$

$$\Rightarrow (d/2)^2 = 20000\mu\Omega cm\frac{1m}{\pi 250k\Omega}$$

Note that $\rho = 20000\mu\Omega cm$, the specific resistance per meter is therefore 100 times as small. Hence, $\rho = 200\mu\Omega m$. Using this we find:

$$\Rightarrow (d/2)^2 = 200\mu\Omega m\frac{1m}{\pi 250k\Omega}$$

$$\Rightarrow d^2/4 = \frac{200\mu\Omega m^2}{\pi 250k\Omega}$$

$$\Rightarrow d = \sqrt{\frac{800\mu\Omega m^2}{\pi 250\Omega}} = 0.03mm$$

The number of circumferences (lengte over de cirkel) of the wire around the tube is equal to:

$$\frac{270°}{360°}2\cdot\pi\cdot r = \frac{270°}{360°}2\cdot\pi\cdot\frac{D}{2} = \frac{3}{4}\cdot\pi\cdot D$$

with $D$ the average diameter of the tube. This 3/4 part of the tube needs be covered with wire where each wire has a thickness $d$. Hence the number of turns $N$ of the wire to cover these 3/4 of the tube is equal to:

$$\Leftrightarrow \frac{3}{4}\cdot 2\pi\cdot 1cm = N\cdot 0.03mm \Rightarrow N = 1571$$

The resolution is therefore equal to:

$$r = \frac{270°}{N} = \frac{270°}{1571} = 0.2°$$