Sensing, Computing, Actuating
Lecture 3 - Systems and Control

Exercise 1: first-order system - temperature sensor

Systems with a thermal capacity such as a liquid thermometer or a thermocouple require a transfer of heat, Q, from the environment to the sensor in order to show a change in temperature. This change in energy, E, as a function of time is described by the following first-order differential equation:

$$Q = \frac{dE}{dt} = mC_V \frac{dT_s(t)}{dt} = hA_s\left(T_o(t) - T_s(t)\right)$$

, with m the weight of the sensor, C_v the specific heat of the sensor, h the heat transfer coefficient, A_s the contact surface (area) of the sensor, T_o the environmental temperature, en T_s the sensor temperature.

(a) Show that the transfer function of the sensor $T_s(s)/T_o(s)$ is equal to:

$$\frac{T_s(s)}{T_o(s)} = \frac{k}{\tau s + 1}$$

, with $k = 1$ and $\tau = \frac{mC_v}{hA_s}$.

Answer: It holds:

$$mC_V \frac{dT_s(t)}{dt} = hA_s\left(T_o(t) - T_s(t)\right)$$

Laplace transformed:

$$\Rightarrow mC_V s T_s(s) = hA_s\left(T_o(s) - T_s(s)\right)$$

$$\Leftrightarrow \left(mC_V s + hA_s\right) T_s(s) = hA_s T_o(s)$$

$$\Leftrightarrow \frac{T_s(s)}{T_o(s)} = \frac{hA_s}{mC_V s + hA_s}$$

$$\Leftrightarrow \frac{T_s(s)}{T_o(s)} = \frac{1}{\frac{mC_V}{hA_s} s + 1}$$

(b) The response of the sensor to a step function on its input is given by:

$$T_s(t) = k\left(1 - e^{-t/\tau}\right)$$

Assume that the sensor has an initial temperature $T_s(0) = T_i$ when the sensor is suddenly exposed to a constant environmental temperature T_o. Show that the response of the sensor is equal to:

$$T_s(t) = T_o + \left(T_i - T_o\right) e^{-t/\tau}$$

Answer: The response to a step function (as given in the previous question) has a static sensitivity k. This is the temperature when the system is at rest ($t \to \infty$). It should then hold for the sensor that $T_s(\infty) = T_o$, therefore it must hold $k = T_o$.
The response for the step function assumes that $T_s(o) = 0$ °C. In reality, the temperature is T_i. This should be corrected in the response. This correction gives the following result:

$$T_s(t) = \mathcal{L}^{-1}\{T_s(s)\} = T_o\left(1 - e^{-t/\tau}\right) + T_i e^{-t/\tau} = T_o + (T_i - T_o)\, e^{-t/\tau}$$

(c) To determine the time constant τ the sensor is exposed from $t = 0$ to a (constant) environmental temperature. The temperature is measured every 3 seconds. This results in the following series of readings:

Time (s)	0	3	6	9	12	15
Temperature (°C)	20.00	35.54	39.00	39.78	39.95	39.99

What is the time constant τ from this sensor?

Answer: We know from the previous question that:

$$T_s(t) = T_o + (T_i - T_o)\, e^{-t/\tau}$$

Using the measurement value, we know that $T_i = 20.00$ °C and $T_o = 39.99$ °C. You can use an arbitrary other temperature from this series to determine τ.

$$\tau = \frac{-t}{ln\left(\frac{T_s(t)-T_o}{T_i-T_o}\right)}$$

Substituting for $t = 3$ s gives:

$$\tau = \frac{-3}{ln\left(\frac{35.54-39.99}{20.00-39.99}\right)} = 2.00s$$

(d) How large is the dynamic error ϵ_d of this sensor in response to a step function? (*Hint:* $\epsilon_d = \lim_{t\to\infty} T_s(t) - k \cdot T_o(t)$)

Answer: The sensor has no dynamic error, i.e., $\epsilon_d = 0$.

(e) Because of temperature fluctuations in the environment, the environmental temperature T_o changes according to: $T_o(t) = 2.3°\text{C}{\cdot}sin(0.50t) + 39.99°\text{C}$. Assume that the time constant τ is equal to 2.00 s. What is the steady-state output of this sensor $T_s(t)$?

Answer: The system is linear, therefore we can use the superposition of signals to compute the output signal.

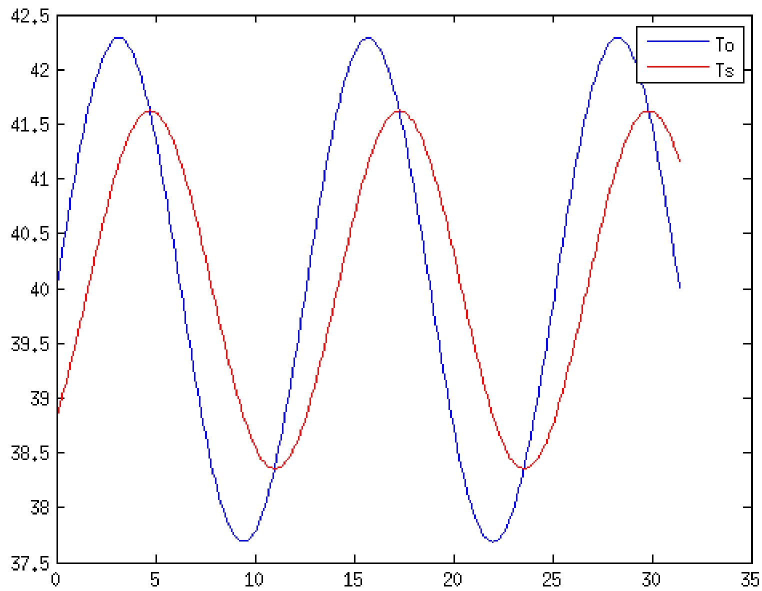
$$\left|\frac{T_s(s)}{T_o(s)}\right|_{\omega=0} = \left|\frac{k}{\sqrt{\omega^2\tau^2+1}}\right| \angle - \arctan(\omega\tau) = 1\angle 0 rad$$

$$\left|\frac{T_s(s)}{T_o(s)}\right|_{\omega=0.50} = \left|\frac{1}{\sqrt{(0.50rad/s)^2(2.00s)^2+1}}\right| \angle - \arctan((0.50rad/s)(2.00s)) = 0.71\angle - 0.79rad$$

The signal at the output of the sensor is therefore equal to:

$$T_s(t) = 39.99 + 2.3 \cdot 0.71 sin(0.50t - 0.79) = 39.99 + 1.63 sin(0.50t - 0.79)$$

Graphically this looks as follows:



The figure clearly shows that the signal at the output of the sensor (measured temperature) correctly follows the environmental temperature. The sensor is therefore usable in this situation.

(f) You want to use the same sensor to measure the temperature of an object whose temperature as a function of time is given by: $T_o(t) = 2.3°\text{C}\cdot sin(20t) + 39.99°\text{C}$. Can you use this sensor to accuratly measure the fast variations in the temperature? (Explain your answer.)

Answer: The system is linear, therefore we can use superposition to compute the output signal.

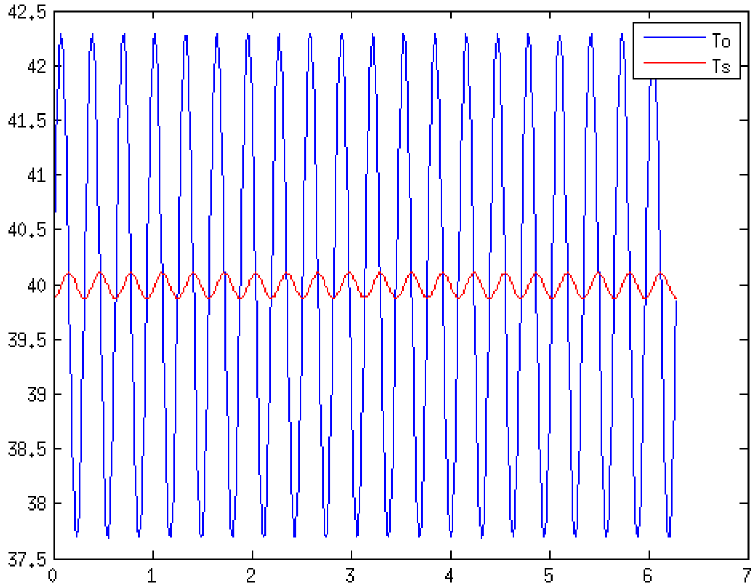
$$\left|\frac{T_s(s)}{T_o(s)}\right|_{\omega=0} = \left|\frac{k}{\sqrt{\omega^2\tau^2+1}}\right| \angle - \arctan(\omega\tau) = 1\angle 0 rad$$

$$\left|\frac{T_s(s)}{T_o(s)}\right|_{\omega=20} = \left|\frac{1}{\sqrt{(20rad/s)^2(2.00s)^2+1}}\right| \angle - \arctan((20rad/s)(2.00s)) = 0.025\angle - 1.55rad$$

The signal at the output of the sensor is therefore equal to:

$$T_s(t) = 39.99 + 2.3 \cdot 0.025 sin(20t - 1.55) = 39.99 + 0.06 sin(20t - 1.55)$$

Graphically this looks as follows:



The figure clearly shows that the fast temperature variation is not clearly visible at the output of the sensor. The sensor is therefore **not** usable for this application. You should decrease the time constant τ to make the sensor usable for measuring $T_o(t)$.

Exercise 2: second-order system - acceleration sensor

A one-axis acceleration sensor consists of a mass whose movement can be translated into an electrical signal. This translation can be performed using for example a capacitive or piezo-electric sensor. The electrical principle is not important for this exercise, we will focus on a mechanical model of the device to analyse its operating characteristics. The figure below shows a generic model for such an acceleration sensor. The mass M is supported by a spring with a spring constant k and the movement of the mass is dampened with a damper that has a damping factor b. The mass may only be moved along the x-axes with respect to the acceleration sensor body. During its use, the sensor is exposed to an acceleration d^2y/dt^2 and the output signal is proportional to the displacement x_0 of the mass M.

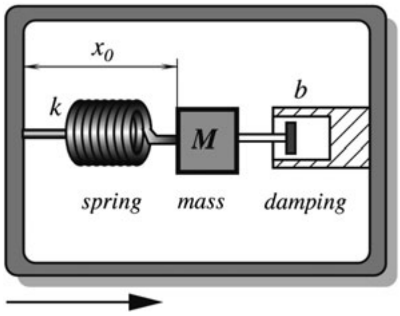
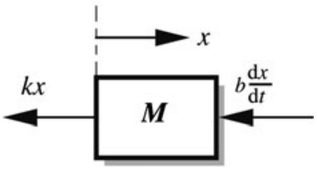


Figure 1: Mechanical model of an acceleration sensor.

(a) Show that the transfer function (in terms of the displacement of the mass $x(t)$ (output) en displacement of the sensor body $y(t)$ (input)) is equal to:

$$\frac{X(s)}{Y(s)} = \frac{M}{k}\frac{(k/M)s^2}{s^2+(b/M)s+k/M}$$

Answer:



For the forces operating on the mass M holds:

$$M\left(\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\right) = kx + b\frac{dx}{dt}$$

The term $\left(\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\right)$ is the difference in acceleration of the armature (body) and mass. Laplace transformed:

$$M\left(s^2Y(s) - s^2X(s)\right) = kX(s) + bsX(s)$$

$$\Leftrightarrow Ms^2Y(s) = \left(Ms^2 + bs + k\right)X(s)$$

$$\Leftrightarrow \frac{X(s)}{Y(s)} = \frac{s^2}{s^2+(b/M)s+(k/M)}$$

$$\Leftrightarrow \frac{X(s)}{Y(s)} = \frac{M}{k}\frac{(k/M)s^2}{s^2+(b/M)s+k/M}$$

(b) Show that the transfer function of the sensor (in terms of the acceleration $a(t)$) is equal to:

$$\frac{X(s)}{A(s)} = \frac{M}{k}\frac{(k/M)}{s^2 + (b/M)s + k/M}$$

Answer: It holds:

$$a(t) = \frac{d^2y(t)}{dt^2}$$

The Laplace transformed is equal to:

$$A(s) = s^2Y(s) \Leftrightarrow Y(s) = \frac{A(s)}{s^2}$$

Substituting this in the solution from the previous question gives:

$$\frac{X(s)}{A(s)/s^2} = \frac{M}{k}\frac{(k/M)s^2}{s^2 + (b/M)s + k/M}$$

$$\Leftrightarrow \frac{X(s)}{A(s)} = \frac{M}{k}\frac{(k/M)}{s^2 + (b/M)s + k/M}$$

(c) Assume that the spring constant k is equal to 508.62 N/m and the mass M has a weight 4.313×10^{-6} kg. Show in a graph the relation between acceleration (x-as) and the displacement of the mass (y-as) over the range from 0 'g' till 30 'g'.

Answer: Hooke's law gives the relation between deflection (X), force and the spring constant:

$$X = \frac{F}{k}$$

The first law of Newton gives the relation between force and acceleration:

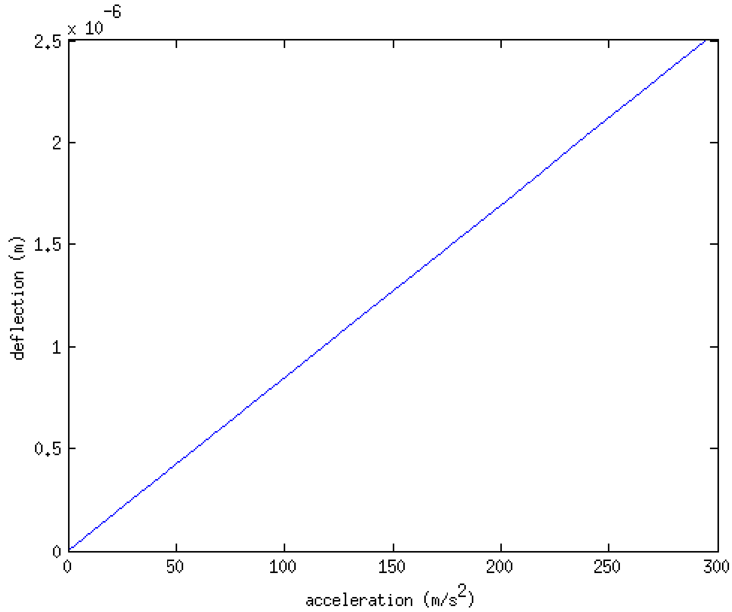
$$F = M \cdot a$$

It therefore holds:

$$X = \frac{M}{k} \cdot a$$

Note that $a = g \cdot 9.81m/s^2$, it therefore holds:

$$X = \frac{M}{k} \cdot 9.81m/s^2 \cdot g$$



(d) Use the values of k and M from the previous question and assume further that the damping factor b is equal to 0.047 Ns/m. You want to use the sensor to measure the displacement of an object of which its position around the centre position shows a sinusoidal movement with a frequency of 0.001 Hz. Is the sensor usable for this application? (Explain your answer.)

Answer: The amplitude of the response at this frequency is equal to:

$$\left|\frac{X(s)}{Y(s)}\right| = \frac{M}{k}\left|\frac{(k/M)(j\omega)^2}{(j\omega)^2 + (b/M)(j\omega) + k/M}\right|$$

$$= \frac{M}{k}\frac{(k/M)(2\pi f)^2}{\sqrt{((2\pi f)^2 + k/M)^2 + ((b/M)(2\pi f))^2}}$$

$$\approx \frac{M}{k}\frac{(k/M)(2\pi f)^2}{k/M} = \frac{M}{k}(2\pi f)^2 = 0.0039\frac{M}{k}$$

The sensor behaves like a high-pass filter for the movement. The sensor can therefore not be used to measure this slow displacement.

(e) You want to use this sensor to measure the acceleration of an object of which the accelation varies sinusoidally between -10 ‘g’ and +10 ‘g’ with a frequency of 0.001 Hz. Is this sensor usable for this application? (Explain your answer.)

Answer: The amplitude of the response at this frequency is equal to:

$$\left|\frac{X(s)}{A(s)}\right| = \frac{M}{k}\left|\frac{(k/M)}{(j\omega)^2 + (b/M)(j\omega) + k/M}\right|$$

$$= \frac{M}{k}\frac{(k/M)}{\sqrt{((2\pi f)^2 + k/M)^2 + ((b/M)(2\pi f))^2}}$$

$$\approx \frac{M}{k}\frac{(k/M)}{k/M} = \frac{M}{k}$$

The sensor behaves as a low-pass filter for the acceleration. The sensor is therefore usable to measure slow accelarations.

(f) The static sensitivity of the sensor is defined as M/k. You can improve the static sensitivity by increasing the mass M. Enlarging the mass has however also an impact on the dynamic behaviour of the system. Explain how the spring constant k and the damping factor b should be changed to compensate the effect of the enlarged mass, while still increasing the static sensitivity of the sensor.

Answer: Assume that the spring constant k remains equal, the damping factor should then be enlarged with the same amount as the mass to remove the impact on the damping. By changing the sensitivity of the sensor, the resonance frequency of the sensor will shift. It will become smaller when the static sensitivity is enlarged.

There are three important parameters in a second-order system, namely damping, undamped natural frequency and static sensitivity. These parameters depend on the physical properties of the system.
Damping:

$$\zeta = \frac{b}{2\sqrt{kM}}$$

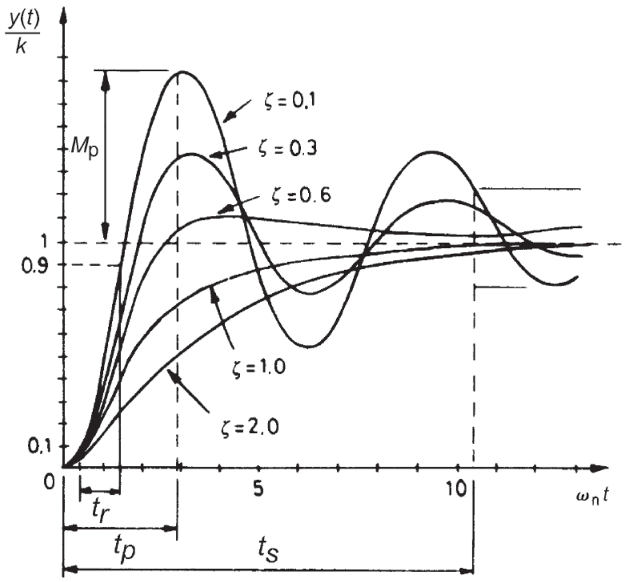
Undamped natural frequency:

$$\omega_n = \sqrt{\frac{k}{M}}$$

Static sensitivity:

$$\frac{M}{k}$$

The parameters together lead to the response on a dynamic input signal. The figure below shows the reponse to a system on which a step function is applied.



There are a few important performance characteristics:
Time needed to go from 10% till 90% of the final output value (rise time):

$$t_r = \frac{\arctan(-\omega_d/\delta)}{\omega_d}$$

, with $\delta = \zeta\omega_n$ (attenuation - damping) and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (undamped natural frequency).

Time till first peak:

$$t_p = \frac{\pi}{\omega_d}$$

Maximal overshoot:

$$M_p = e^{-(\delta/\omega_d)\pi}$$