# Sensing, Computing, Actuating
# Lecture 4 - Bridge circuit / Strain

**Exercise 1: mass airflow sensor**

The mass airflow sensor of an injection engine measures the amount of air that enters the combustion chamber through the air intake. Before injecting the right amount of fuel, the engine needs to know exactly how much air comes into the cylinders. For this purpose, it is more important to know the mass rather then the volume of the air. The density of the air varies depending on the temperature and altitude at which the vehicle is operated. To measure the air mass, you can use a so-called "hot wire mass airflow sensor" (see Figure 1). In this sensor, the air is heated using a hot wire. In the case of non-moving air, the air on both sides of the heating element will be heated to the same temperature. Both RTDs ($R_3$ and $R_4$) measure in that case the same temperature[1]. As soon as the air starts moving, the temperature will rise at one RTD, whereas it lowers on the other RTD. The change in temperature is related to the density and flow-speed of the air (and therefore with the mass of the air).

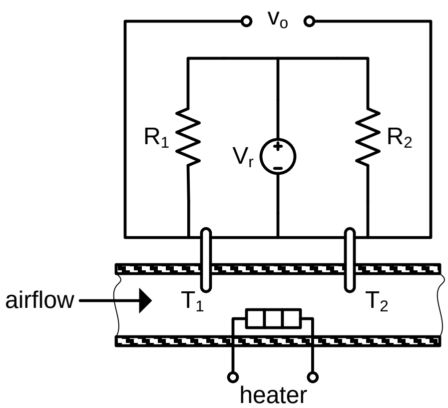



Figure 1: Mass airflow sensor.

The circuit in Figure 2 is used to measure a temperature difference between 0°C and 100°C in the range of -60°C till +100°C. The maximal error in the measured temperature should be less then 1°C. The resistors $R_3$ en $R_4$ are both temperature dependent resistors (RTDs). Both RTDs are of the same type: PT100 with $R_0 = 100\ \Omega$ and $\alpha = 0.004/°C$ at 0°C. The dissipation constant $\delta$ of the air varies (depending on the air density and flow-speed) between 1 mW/K and 4 mW/K.

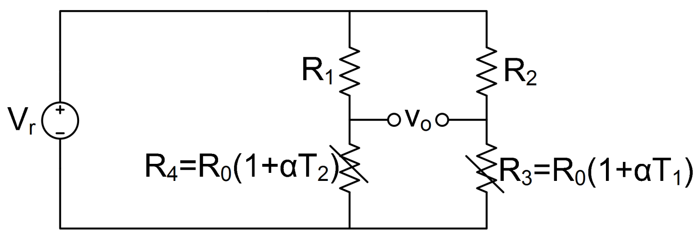



Figure 2: Electrical equivalent circuit of mass airflow sensor.

[1] In a practical implementation it is possible to integrate the heater with one of the RTDs.

(a) Show that the output voltage of the bridge is equal to:

$$v_o = \frac{k\alpha\left(T_1 - T_2\right)}{\left(1 + k + \alpha T_1\right)\left(1 + k + \alpha T_2\right)} V_r$$

with $k = R_1/R_0 = R_2/R_0$.

**Answer:**

$$v_o = \left(\frac{R_3}{R_2 + R_3} - \frac{R_4}{R_1 + R_4}\right) V_r$$

$$\Leftrightarrow v_o = \left(\frac{R_0 + R_0\alpha T_1}{R_2 + R_0 + R_0\alpha T_1} - \frac{R_0 + R_0\alpha T_2}{R_1 + R_0 + R_0\alpha T_2}\right) V_r$$

$$\Leftrightarrow v_o = \left(\frac{1 + \alpha T_1}{k + 1 + \alpha T_1} - \frac{1 + \alpha T_2}{k + 1 + \alpha T_2}\right) V_r$$

$$\Leftrightarrow v_o = \frac{k\alpha\left(T_1 - T_2\right)}{\left(1 + k + \alpha T_1\right)\left(1 + k + \alpha T_2\right)} V_r$$

(b) Show that the absolute error in the temperature difference due to the non-linearity of the circuit is equal to:

$$\epsilon = \left|\left(T_2 - T_1\right)\frac{\alpha\left(T_1 + T_2\right)(k+1) + \alpha^2 T_1 T_2}{\left(1 + k + \alpha T_1\right)\left(1 + k + \alpha T_2\right)}\right|$$

**Answer:** So far we computed the absolute error using the following equation:

$$\epsilon = |v_o - v_{o,ideal}|$$

, with $v_o$ the measured output voltage and $v_{o,ideal}$ the ideal output voltage (without non-linearity). This gives the absolute error in terms of the output voltage. In this question we need to compute the error in terms of the input signal (i.e., the temperature difference).

$$\epsilon = |\Delta T_{measured} - \Delta T_{real}|$$

$\Delta T_{real}$ is equal to:

$$\Delta T_{real} = T_1 - T_2$$

$\Delta T_{measured}$ must be computed based on the measured output voltage. It holds that:

$$\Delta T_{measured} = \frac{v_o}{S}$$

, with $v_o$ the measured output voltage (including the non-linearity) and $S$ the sensitvity of the sensor. To determine the sensitivity we will assume a linear relation between input (temperature difference) and output voltage. Our assumption is therefore:

$$v_{o,ideal} = S \cdot \Delta T = S \cdot \left(T_1 - T_2\right)$$

The ideal output voltage is therefore equal to:

$$v_{o,ideal} = \frac{k\alpha\left(T_1 - T_2\right)}{(1+k)(1+k)} V_r$$

You can determine this value by ignoring the non-linearity ($\alpha T_1$ en $\alpha T_2$) in the denominator of $v_o$. The sensitivity $S$ is thus equal to:

$$S = \frac{v_{o,ideal}}{(T_1 - T_2)} = \frac{k\alpha V_r}{(1+k)^2}$$

The error in terms of the input signal (temperature difference) is thus equal to:

$$\epsilon = \left|\frac{v_o}{S} - (T_1 - T_2)\right| = \left|\frac{(1+k)^2 (T_1 - T_2)}{(1+k+\alpha T_1)(1+k+\alpha T_2)} - (T_1 - T_2)\right|$$

$$\Leftrightarrow \epsilon = \left|(T_2 - T_1)\frac{\alpha (T_1 + T_2)(k+1) + \alpha^2 T_1 T_2}{(1+k+\alpha T_1)(1+k+\alpha T_2)}\right|$$

(c) What values should the resistors $R_1$ and $R_2$ have such that the maximal error is always smaller then 0.1°C? (Hint: you may ignore the self-heating error in this question.)

**Answer:**
The error increases with an increasing temperature difference and with the absolute temperature. The worst-case situation occurs when $T_1 = 0$°C and $T_2 = 100$°C. It must hold that:

$$\epsilon = \left|(100^\circ C)\frac{0.004\,(100)\,(k+1) + (0.004)^2(0)(100)}{(1+k+(0.004)(0))\,(1+k+(0.004)(100))}\right| = 0.1^\circ C$$

Solving this equation yields: k = 399. It holds that $R_0 = 100\Omega$.

Therefore it must hold that $R_1 = R_2 = 39.9k\Omega$.

(d) Which value should the supply voltage have such that the error due to self-heating is always less then 0.01°C? (Hint: $\Delta T = P_D/\delta = \left(I^2 \cdot R\right)/\delta$)

**Answer:**
The error due to self-heating is equal to:

$$\epsilon_s = \left(\frac{V_r}{R_1 + R_4}\right)^2 \frac{R_4}{\delta} - \left(\frac{V_r}{R_2 + R_3}\right)^2 \frac{R_3}{\delta} = \frac{V_r^2}{R_0 \delta}\left(\frac{1+\alpha T_2}{(1+k+\alpha T_2)^2} - \frac{1+\alpha T_1}{(1+k+\alpha T_1)^2}\right)$$

Since $k$ is large, the self-heating error can be approximated as follows:

$$\epsilon_s \approx \frac{V_r^2}{R_0 \delta}\left(\frac{1+\alpha T_2}{(1+k)^2} - \frac{1+\alpha T_1}{(1+k)^2}\right) = \frac{V_r^2}{R_0 \delta}\frac{\alpha (T_2 - T_1)}{(1+k)^2}$$

The error $\epsilon_s$ should always be less then 0.01°C. The maximal error is reached at the lowest dissipation constant. To meet the requirement at that point, the supply voltage should be equal to $V_r = 20.0$ V.

**Exercise 2: manifold absolute pressure sensor**

The manifold absolute pressure (MAP) sensor measures the absolute pressure in the inlet manifold of a fuel injected engine. The sensor generates a signal proportional to the absolute pressure in the inlet manifold. The engine's electronic control unit (ECU) uses this information to determine the ignition timing and the appropriate mixture of the fuel to optimize the combustion.

If the gas pedal is pressed causing the engine to work hard, the air pressure will rise in the engine. In that situation, the engine needs to take in more air. More fuel should also be supplied to the engine in order to keep the air/fuel ratio nearly constant. To be precise, the ECU will make enrich fuel mixture a bit when the computer measures a heavy load. In this way, the engine is able to deliver a bit more power when needed. At the same time, the ECU will delay the ignition timing of the engine to prevent damage to its internals.

The engine needs to supply less power when the car is moving at a constant speed. In that situation, the driver will apply less pressure on the gas pedal. The will cause a reduction in the air pressure inside the engine. The MAP sensor detects this and the ECU will respond to this by adjusting the fuel mixture ensuring that the car cruises most economically.

A MAP sensor can be realized using a number of strain gauges. Figure 3 shows the cross section of such a MAP sensor. The sensor consists of a vacuum hollow room. At the top of the room, a membrane is attached that changes shape when (air)pressure is applied. By attaching one or more strain gauges to the member, it becomes possible to measure the change in the dimensions of the membrane. Depending on the placement of the strain gauges, they are compressed or expanded when pressure is applied to the membrane.

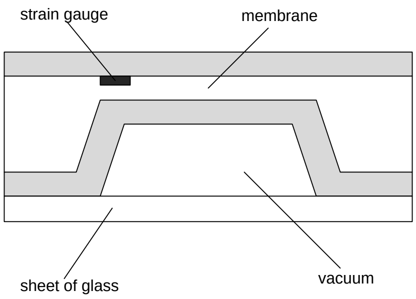



Figure 3: MAP sensor based on membrane and strain gauges.

Figure 4 shows the electrical equivalent circuit of such a MAP sensor with two strain gauges. In addition to the strain gauges, the circuit contains also two fixed resistors. The Young's modulus of the membrane, $E$, is equal to $210 \cdot 10^9$ N/m$^2$. When unloaded, all four resistors have a resistance of 350 Ω. The strain gauges have a gauge factor of 2.10. To prevent any damage to the strain gauges, the current through them should be limited to 10 mA.

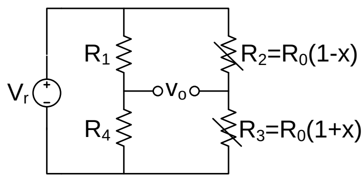



Figure 4: Electrical equivalent circuit of the MAP sensor.

(a) What is the maximal voltage, $V_r$, that can be placed over the bridge?

**Answer:** The total resistance from the bridge, as seen by the voltage supply, is equal to:

$$R_{tot} = (R_1 + R_4)//(R_2 + R_3) = \frac{(R_1 + R_4)(R_2 + R_3)}{(R_1 + R_4) + (R_2 + R_3)} = \frac{4R_0^2}{4R_0} = R_0 = 350\Omega$$

The maximal current through a strain gauge is limited to 10 mA. The maximal current from the source is thus equal to 20 mA (10 mA left and 10 mA right). The maximal voltage across the bridge is thus equal to: $V_r = R_{tot} \cdot I_{max} = 350\ \Omega \cdot 20$ mA $= 7$ V.

(b) Assume that the supply voltage of the bridge ($V_r$) is equal to 7 V. What is the output voltage of the bridge ($v_o$) when a pressure of 9.8 MPa is applied to the membrane?

**Answer:**
The relation between stress ($\sigma$), strain ($\epsilon$) and Young's modulus ($E$):

$$\sigma = \frac{F}{a} = E\epsilon = E\frac{dl}{l}$$

A pressure of 9.80 MPa corresponds to a stress of $9.80 \cdot 10^6$ N/m$^2$.
The strain is therefore equal to:

$$\epsilon = \frac{\sigma}{E} = \frac{9.80 \cdot 10^6 N/m^2}{210 \cdot 10^9 N/m^2} = 4.67 \cdot 10^{-5}$$

The gauge factor gives the relation between strain and the change in resistance:

$$\frac{dR}{R} = G\frac{dl}{l} = G\epsilon = 2.10 \times 4.67 \cdot 10^{-5} = 9.80 \cdot 10^{-5}$$

The change in resistance is thus equal to:

$$dR = R \times 9.80 \cdot 10^{-5} = 350\Omega \times 9.80 \cdot 10^{-5} = 0.0343\Omega$$

This change in resistance occurs in two of the four strain resistors. In one strain gauge, the resistance decreases with this amount and in another strain gauge it increases with this amount. The output voltage of the bridge is therefore equal to:

$$v_o = \left(\frac{R_4}{R_1 + R_4} - \frac{R_3}{R_2 + R_3}\right) V_r$$

$$\Rightarrow v_o = \left(\frac{R_0}{R_0 + R_0} - \frac{R_0 + dR}{R_0 - dR + R_0 + dR}\right) V_r$$

$$\Leftrightarrow v_o = \left(\frac{R_0}{2R_0} - \frac{R_0 + dR}{2R_0}\right) V_r$$

$$\Leftrightarrow v_o = -\frac{dR}{2R_0} V_r = \frac{-0.0343\Omega}{2 \cdot 350\Omega} \cdot 7V = -3.43 \cdot 10^{-4} V$$

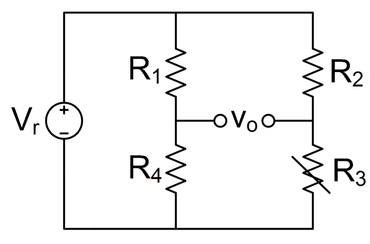



Figure 5: Bridge with one strain gauge.

(c) The use of a bridge with two strain gauges might be to expensive from some applications. Therefore, the circuit shown in Figure 4 can be replaced with a circuit that contains only one strain gauge (see Figure 5). This strain gauge has a resistance of 350 Ω when unloaded and a gauge factor of 2.10. The resistance of this strain gauge is given by $R_3 = R_0(1 + x)$. The maximal pressure that can be applied to this strain gauge is 98.0 MPa. The strain gauge is used in a bridge circuit with three fixed resistors (see Figure 5). The resistors $R_1$, $R_2$ and $R_4$ all have a resistance of 350 Ω. The supply voltage of the bridge, $V_r$, is equal to 7 V. The output voltage of this circuit will show a non-linearity because the circuit contains only one strain gauge. What is the maximal relative error when the output voltage of the bridge ($v_o$) is assumed to have a linear dependency on the stress $y$ with a sensitivity equal to the sensitivity at $y = 0$?

**Answer:** The measured output voltage of the bridge is equal to:

$$v_o = \left(\frac{R_4}{R_1 + R_4} - \frac{R_3}{R_2 + R_3}\right) V_r$$

$$\Rightarrow v_o = \left(\frac{R_0}{R_0 + R_0} - \frac{R_0(1 + x)}{R_0 + R_0(1 + x)}\right) V_r$$

$$\Rightarrow v_o = \left(\frac{1}{2} - \frac{1 + x}{2 + x}\right) V_r$$

$$\Rightarrow v_o = \left(\frac{2 + x - 2(1 + x)}{2(2 + x)}\right) V_r$$

$$\Rightarrow v_o = \frac{-x}{4 + 2x} V_r$$

The ideal output voltage of the bridge is equal to

$$v_{o,ideaal} = \frac{-x}{4} V_r$$

You will find this ideal output voltage when ignoring the non-linearity (ignore $x$ in the denominator). This question assumes that the output voltage of the bridge ($v_o$) has a linear dependency on the stress $y$ with a sensitivity equal to the sensitivity at $y = 0$. At $y = 0$ holds $x = 0$. The sensitivity at $x = 0$ is equal to:

$$\left.\frac{dv_o}{dx}\right|_{x=0} = \left.\frac{-4 - 2x + 2x}{(4 + 2x)^2}\right|_{x=0} = \left.\frac{-4}{(4 + 2x)^2}\right|_{x=0} = -\frac{1}{4}$$

Using this approach, you also find:

$$v_{o,ideaal} = -\frac{1}{4} x V_r = \frac{-x}{4} V_r$$

The relative error due to the non-linearity is thus equal to:

$$\epsilon = \left|\frac{v_o - v_{o,ideaal}}{v_{o,ideaal}}\right| = \left|\frac{v_o}{v_{o,ideaal}} - 1\right| = \left|\frac{4}{4 + 2x} - 1\right| = \left|\frac{4 - 4 - 2x}{4 + 2x}\right| = \left|\frac{-2x}{4 + 2x}\right| = \left|\frac{-x}{2 + x}\right|$$

It holds: $x \geq 0$, thus:

$$\epsilon = \frac{x}{2 + x}$$

Determine now the point at which the maximal non-linearity occurs:

$$\frac{d\epsilon}{dx} = 0$$

$$\Rightarrow \frac{d\epsilon}{dx} = \frac{2 + x - x}{(2 + x)^2} = \frac{2}{(2 + x)^2}$$

The maximal non-linearity occurs when $x$ reaches the end of its range. This occurs when the pressure is at its maximum. Hence, the maximal non-linearity occurs when $y = 98.0$ MPa. The change in resistance is in this situation equal to: $0.343\Omega$ (see question (2b)). This implies:

$$x = \frac{dR}{R_0} = \frac{0.343\Omega}{350\Omega} = 9.80 \cdot 10^{-4}$$

The maximal relative error due to the non-linearity is therefore equal to:

$$\epsilon = \frac{9.80 \cdot 10^{-4}}{2 + 9.80 \cdot 10^{-4}} \cdot 100\% = 0.049\%$$

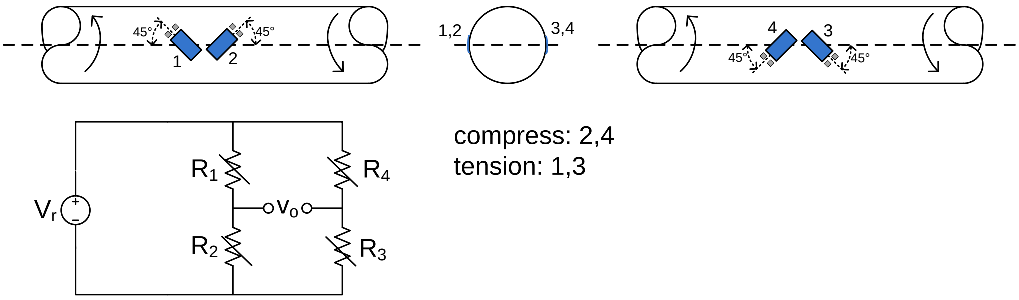
(d) The resistance of a strain gauge changes when it is stretched. This change in resistance is caused by two effects. Name both effects and explain which effect is important in metals and in silicon strain gauges.

**Answer:**

- Change in dimensions (metals)
- Change in relative conductivity (silicon)

(e) Strain gauges are used amongst others to measure torque. Assume that you want to measure the torque on a cylindrical tube using strain gauges. Where should the strain gauges be placed and why?

**Answer:**




Two strain gauges are compressed and two are stretched when the tube is twisted due to the moment (torque).