Introduction into Network Calculus

Stephan Recker
stephan.recker@vodafone.com

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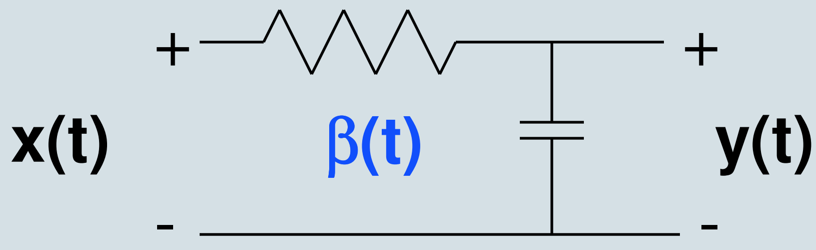
Overview

1. What is Network Calculus?
2. Arrival Curves
3. Min-Plus Convolution
4. Min-Plus Deconvolution
5. Service Curves
6. Single Server Analysis
7. Multiple Server Analysis – Composition Theorem

The Standard Linear Theory



LTI filter in conventional algebra (R, +, ×)

- Input signal = electrical voltage x(t)
- System = circuit (filter) with impulse response β(t)
- Output = convolution of x(t) and β(t) :

$$y(t) = \int \beta(t - s)x(s)ds$$

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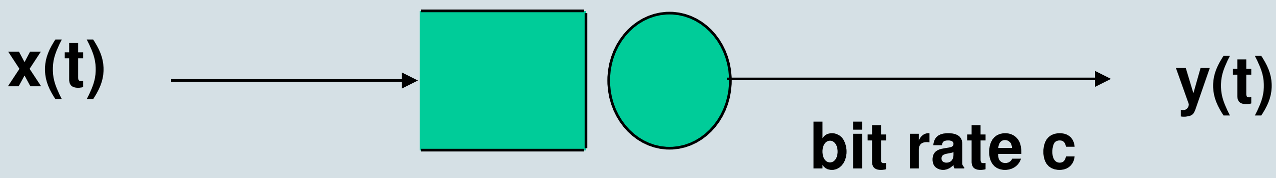
Network Calculus uses Min-Plus Linear Theory



A linear system in min-plus algebra (R, min, +)

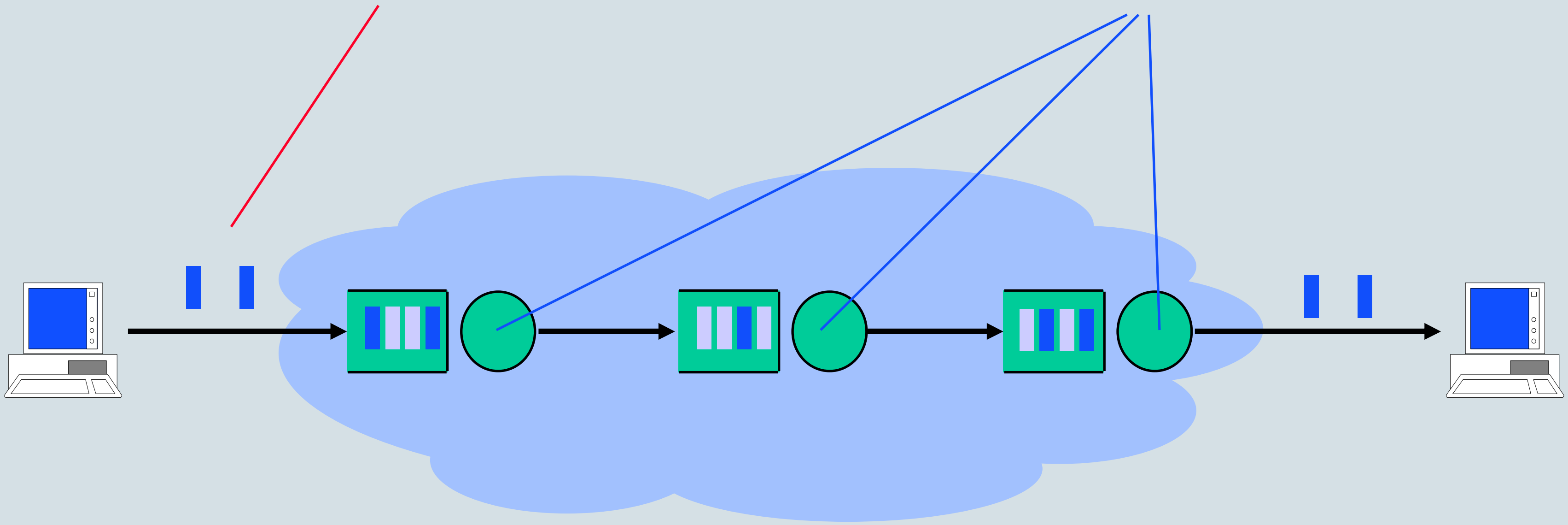
- Input = arrived traffic in [0,t]: x(t)
- System = server with rate c : β(t) = ct
- Output = convolution of x(t) and β(t):

$$y(t) = \inf_{s \in \mathbb{R}} \{\beta(t-s) + x(s)\}$$

$$\beta(t), x(t) = 0 \ \ \forall \ t < 0$$

Two Key Concepts: Arrival and Service Curve

IntServ and DiffServ use the concepts of *arrival curve* and *service curves*



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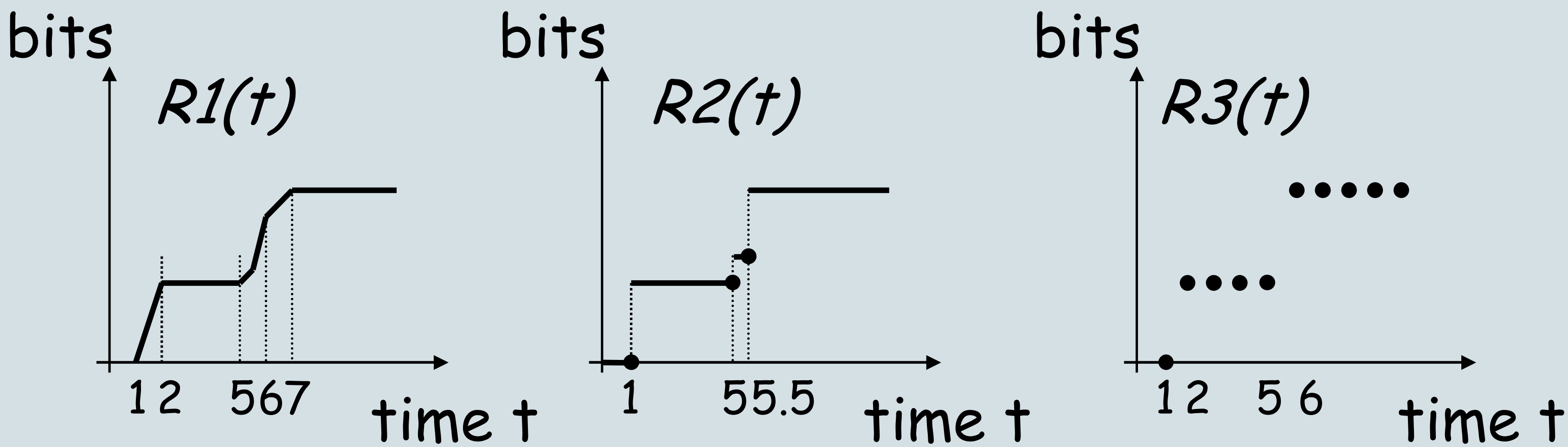
Cumulative Flows

Cumulative flow $R(t) \in \mathcal{F}$, *t* real or integer

$$\mathcal{F} = \{x(t) \text{ is non decreasing and } x(t) = 0 \ \forall \ t < 0\}$$

Examples:



Fluid model (continuous) Packet model Discrete-time model

Arrival Curves

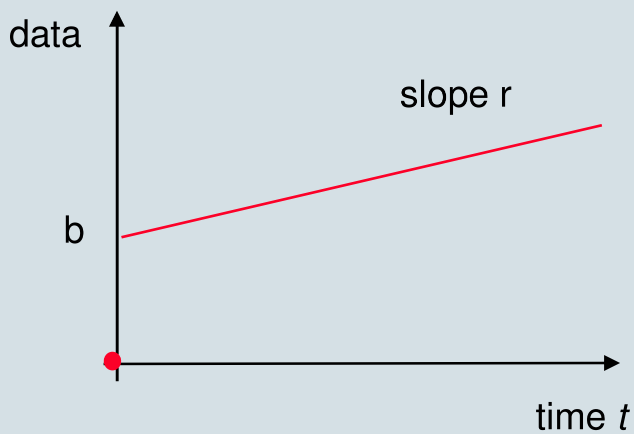
Arrival curve $\hat{A}(t)$:

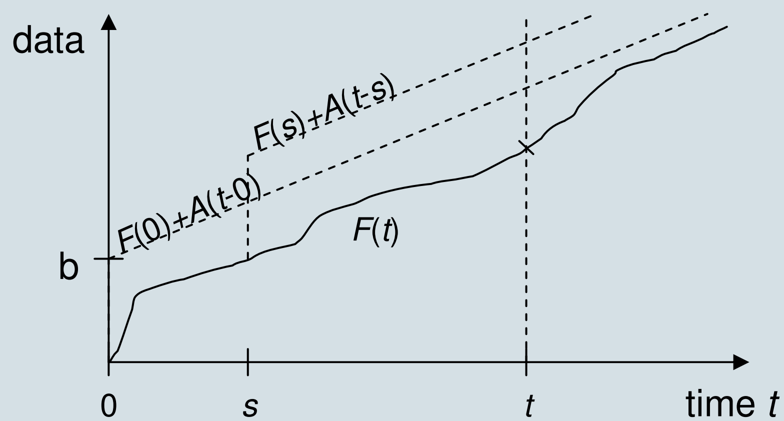
For any times $0 \le s \le t$, the cumulative flow R(t) satisfies

$$R(t) - R(s) \le \hat{A}(t - s)$$

Example 1: affine arrival curve $\gamma_{r,b}$

$$\hat{A}(t) = \gamma_{r,b}(t) = r \cdot t + b \quad \forall \quad t > 0$$





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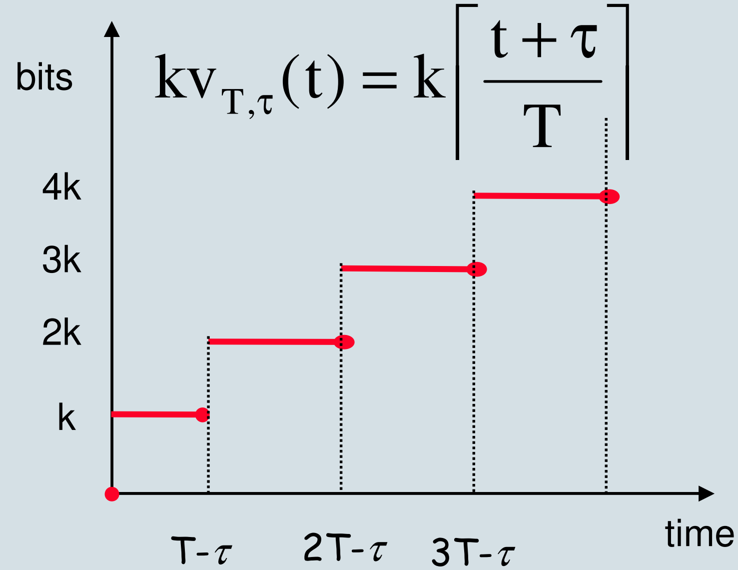
Arrival Curves

Example 2: stair arrival curve $kv_{T,\tau}$

- $\hat{A}(t) = kv_{T,\tau}(t) = k\left\lceil \frac{t+\tau}{T} \right\rceil$
  with T = period, τ = tolerance, k = constant packet size
- Characterizes flows that are periodic stream of packets of same size *k* (cells), that suffers a variable delay <= τ

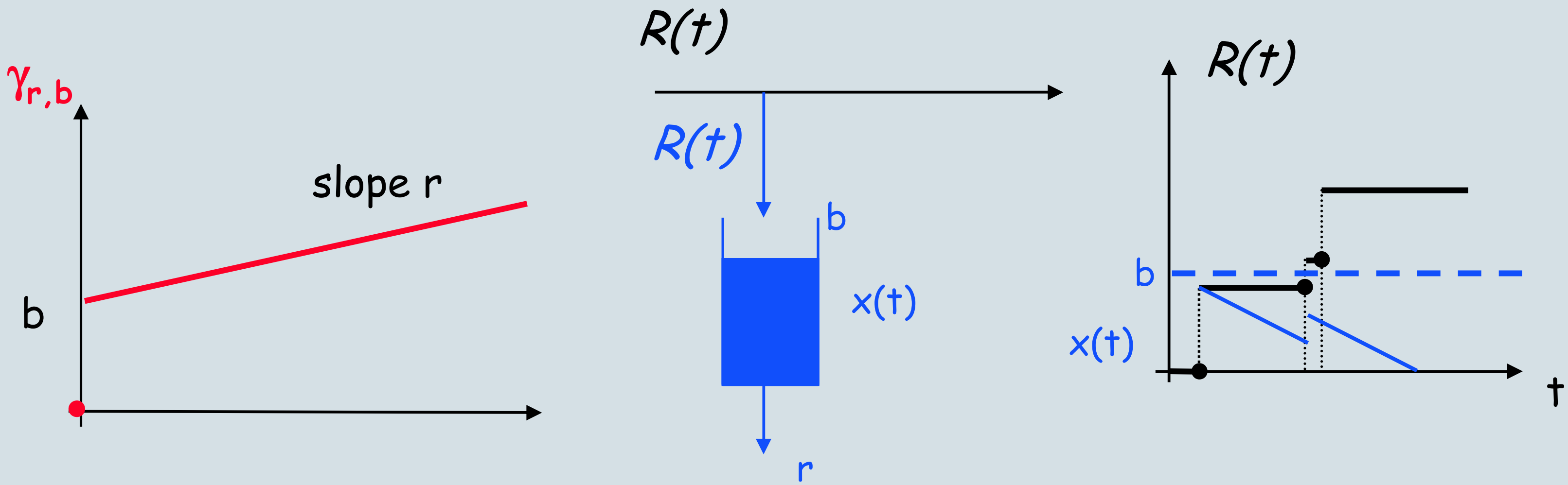
All packets of size k.
If R conforms to $kv_{T,\tau}$
then R conforms to $\gamma_{r,b}$
with r = k/T and
b = k(τ+T)/T



Leaky Bucket

All packets of flow R are declared conformant by a leaky buket controller of rate r and size b

⇔ *R* conforms to $\hat{A}(t) = \gamma_{r,b}(t) = r \cdot t + b \quad \forall \; t > 0$



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Arrival Curves and Min-Plus Convolution

Arrival Curve property
means for all $0 \leq s \leq t$:

$$x(t) - x(s) \leq \hat{A}(t-s)$$

$$\Leftrightarrow x(t) \leq x(s) + \hat{A}(t-s) \quad \forall \;\; 0 \leq s \leq t$$

$$\Leftrightarrow x(t) \leq \inf_{0 \leq u \leq t} \left[x(u) + \hat{A}(t-u) \right]$$

$$\Leftrightarrow x(t) \leq \inf_{u \in \mathbb{R}} \left[x(u) + \hat{A}(t-u) \right] \quad \text{with} \;\; x(t), \hat{A} \in \mathcal{F}$$

$$\Leftrightarrow x(t) \leq \left(x \otimes \hat{A} \right)(t)$$

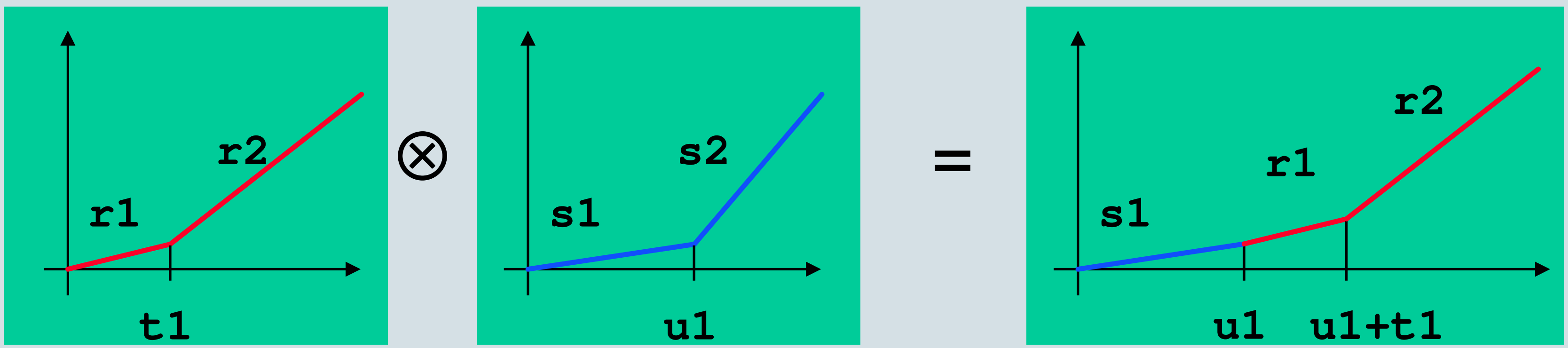
Min-Plus Convolution

Properties of Min-Plus Convolution

- $(f \otimes g) \in \mathcal{F}$
- $\otimes$ is associative
- $\otimes$ is commutative
- Neutral element: δ_0 : $f \otimes \delta_0 = f$ $\delta_0(t) = 0$ for $t = 0$ and $\delta_0(t) = \infty$ for $t > 0$
- $\otimes$ is distributive with respect to min ($\wedge$)
- Functions passing through the origin ($f(0) = g(0) = 0$): $f \otimes g \leq f \wedge g$
- Concave functions passing through the origin: $f \otimes g = f \wedge g$
- Convex piecewise linear functions: $f \otimes g$ is the convex piecewise linear function obtained by putting end-to-end all linear pieces of f and g, sorted by increasing slopes

Min-Plus Convolution of Convex Functions

convex piecewise linear wide-sense increasing, passing by origin: put segments end to end with increasing slope



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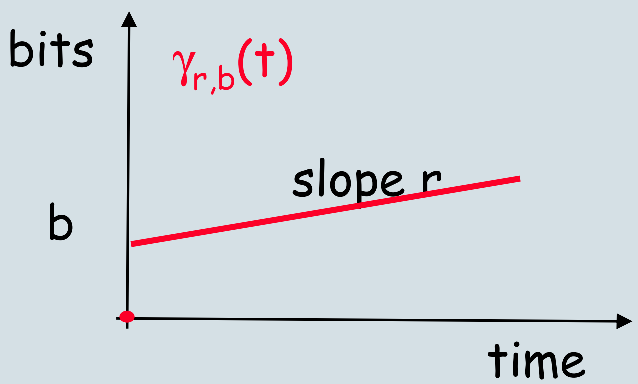
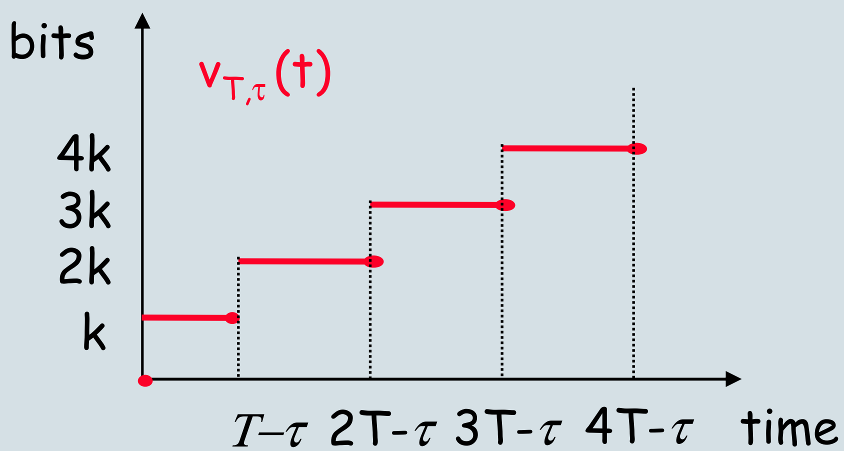
Sub-Additive Functions

- f is sub-additive ⇔ f (t) + f(s) ≥ f(t+s)
- f is concave with f(0) = 0 ⇒ f is sub-additive
- f,g are sub-additive and pass through the origin: f(0) = g(0) = 0 ⇒ f ⊗ g is sub-additive



$\gamma_{r,b}$ is concave



$v_{T,\tau}$ is not concave, but is sub-additive

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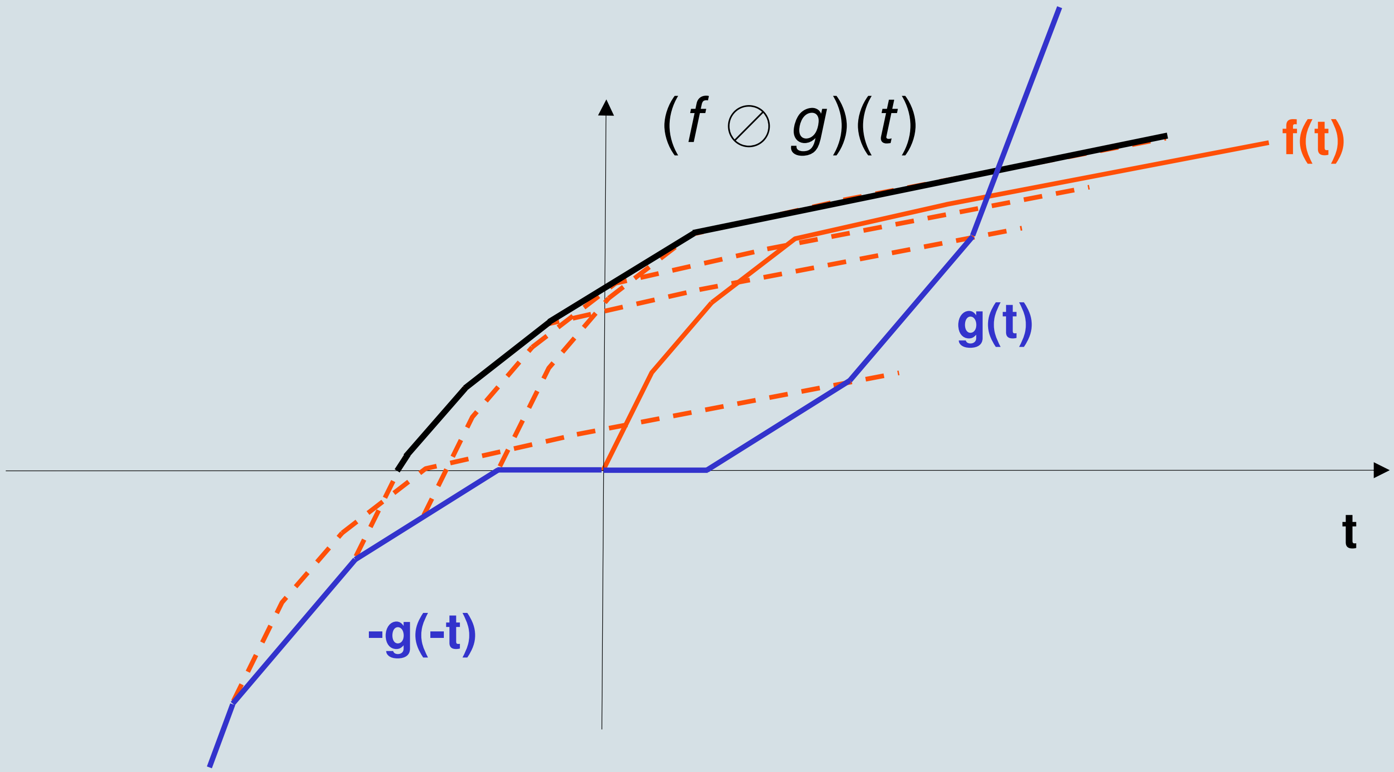
Min-Plus Deconvolution

Definition: $h(t) \le (f \oslash g)(t) = \sup_u \left[f(t+u) - g(u) \right]$



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Properties of Min-Plus Deconvolution

$(f \oslash g) \notin \mathcal{F}$ in general

$(f \oslash f)$ is sub-additive with $(f \oslash f)(0) = 0$

$(f \oslash f) \in \mathcal{F}$

$(f \oslash g) \oslash h = f \oslash (g \otimes h)$

Duality with $\otimes$: $f \oslash g \leq h \Leftrightarrow f \leq g \otimes h$

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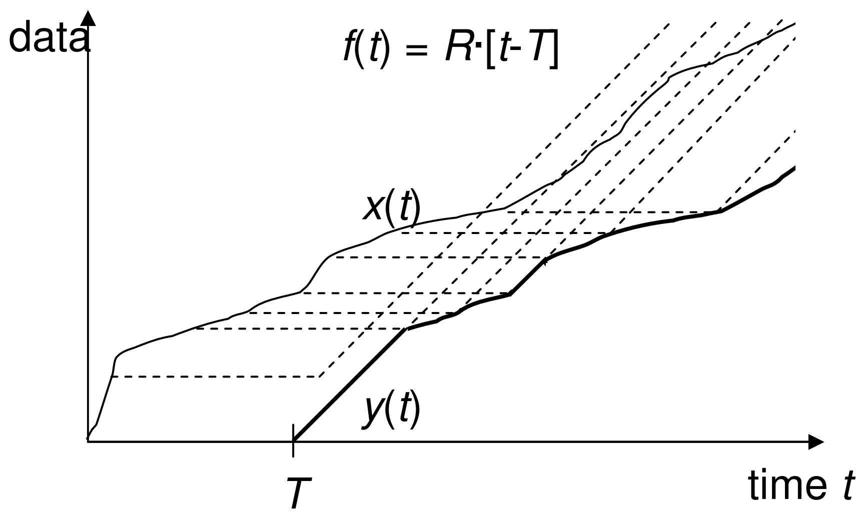
Service Curves



Service curve f(t) : Minimum service offered.

$$y(t) \geq y(s) + f(t - s)$$

in case of backlog

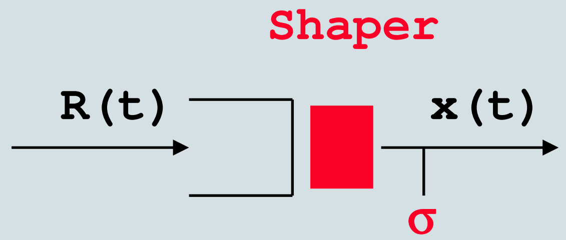
For all t there exists at least one $s \leq t$, such that:

$$y(t) \geq x(s) + f(t - s)$$

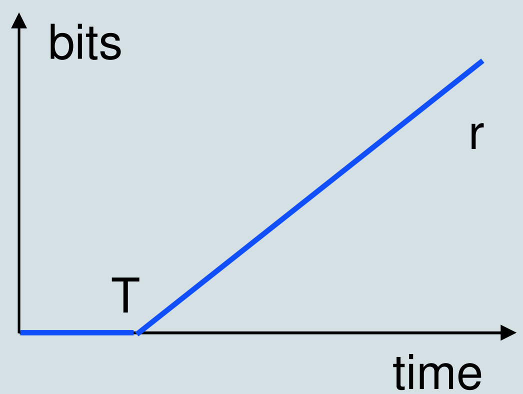
(i.e. s is the start of the last backlogged period)

$$y(t) \geq \inf_{t \geq s \geq 0} \left[x(s) + f(t - s) \right]$$

Service Curve Example



- If σ is sub-additive and $\sigma(0) = 0 \Rightarrow x(t) = (R \otimes \sigma)(t)$
- The service curve of a shaper is thus f(t)=σ(t).
- Standard Internet Router Model: Rate-Latency



$$f(t) = R(t - T)$$

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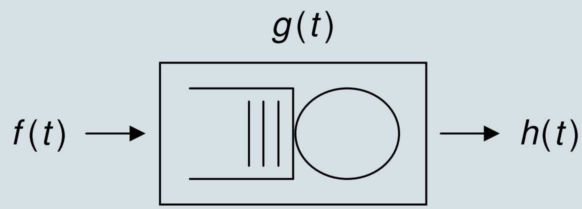
Summary of Network Calculus



Lower Service Bound $h(t) \geq (f \otimes g)(t) = \inf_u \left[f(t-u) + g(u) \right]$

Upper Service Bound $h(t) \leq (f \oslash g)(t) = \sup_u \left[f(t+u) - g(u) \right]$

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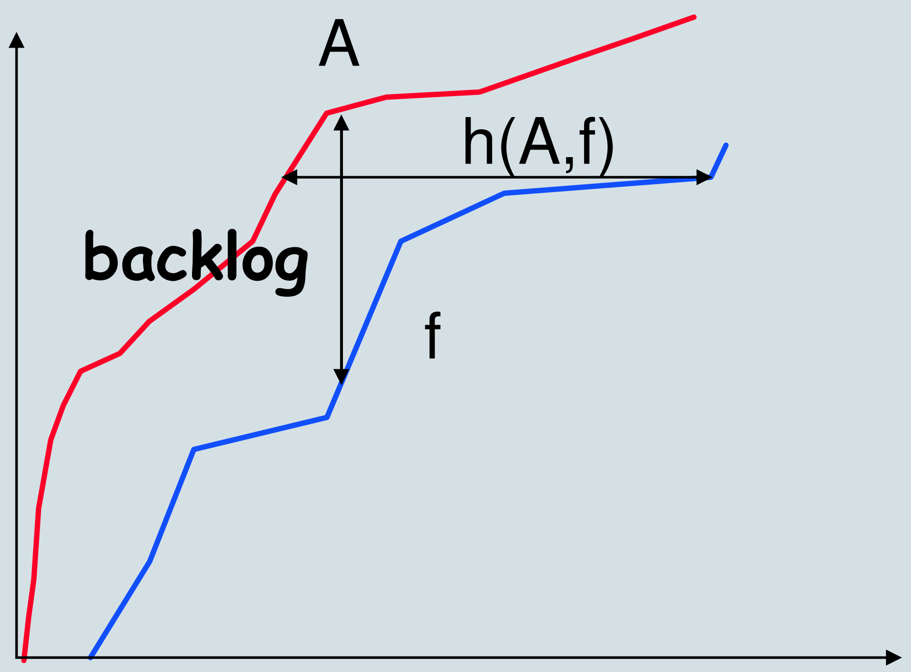
Tight Bounds on Delay and Backlog

- If flow has arrival curve A and node offers service curve f then
- backlog ≤ sup (A(s) -f(s))
- delay ≤ h(A, f)



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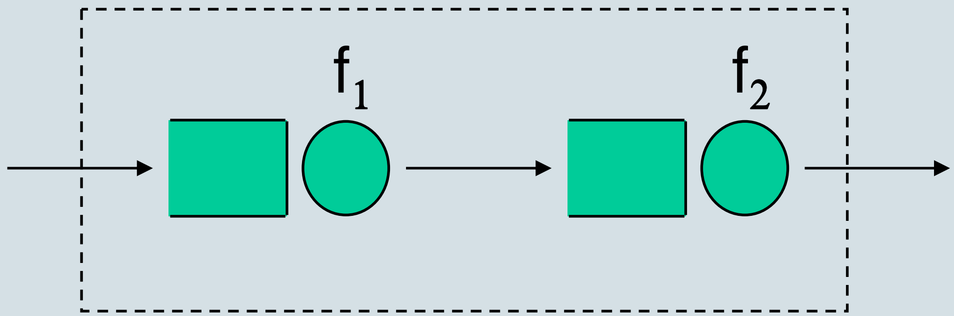
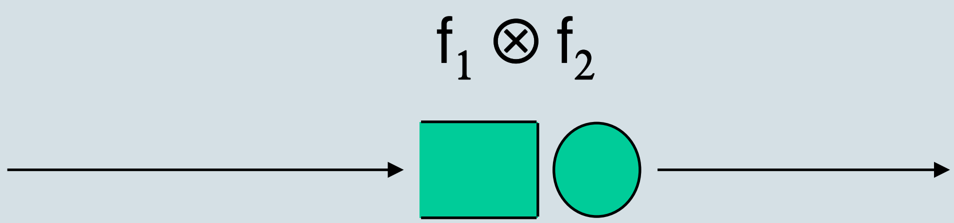
The Composition Theorem

Theorem: the concatenation of two network elements each offering service curve f_i offers the service curve $f_1 \otimes f_2$





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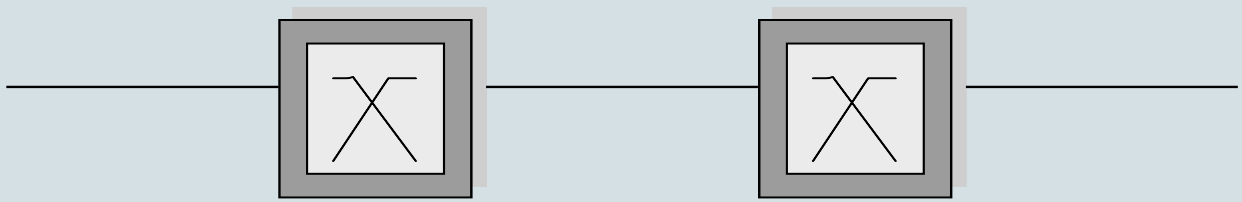
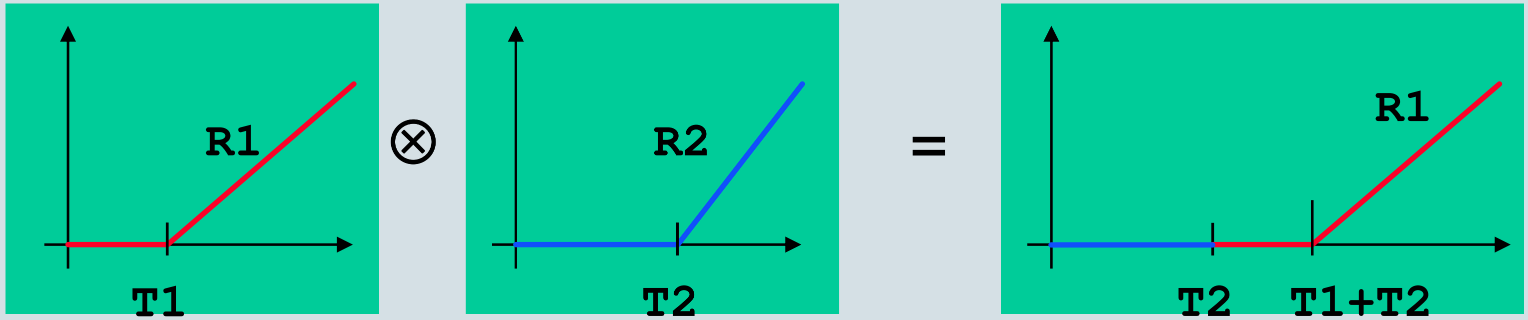
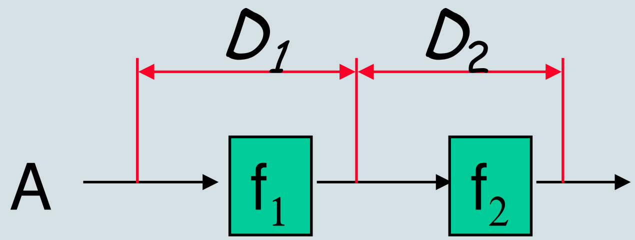
Example: Tandem of Routers

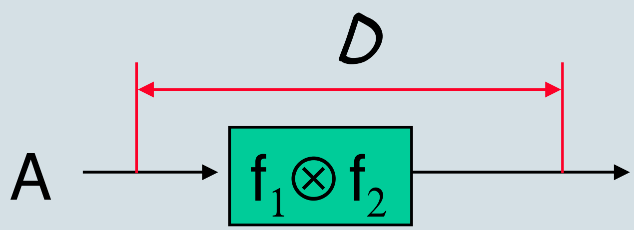




Pay Bursts Only Once



$$D_1 + D_2 \leq (2b + RT_1)/R + T_1 + T_2$$

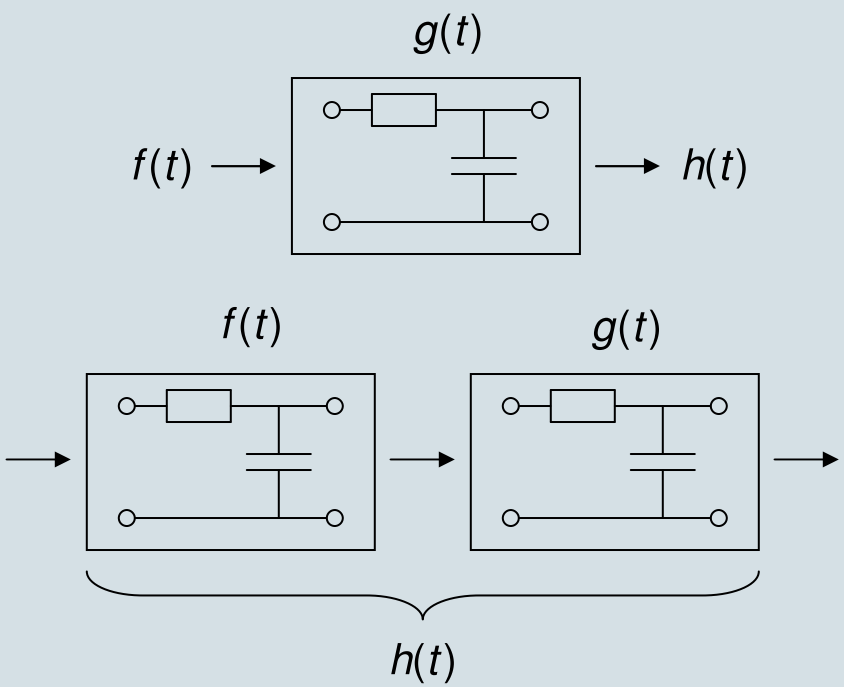


$$D \leq b/R + T_1 + T_2$$

end to end delay bound is less

Summary: Linear Systems and Network Calculus

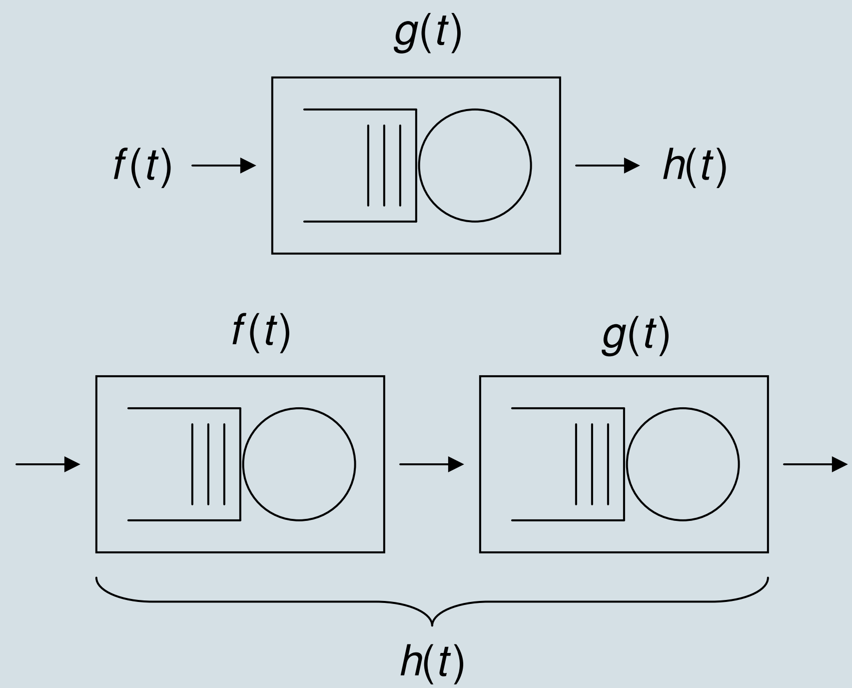
Linear System Theory



Convolution

$$h(t) = (f * g)(t) = \int_{-\infty}^{+\infty} f(t-u)g(u)du$$

Network Calculus



Min-plus convolution

$$h(t) \geq (f \otimes g)(t) = \inf_u [f(t-u) + g(u)]$$

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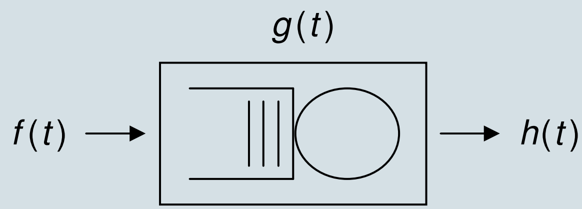
Summary of Network Calculus



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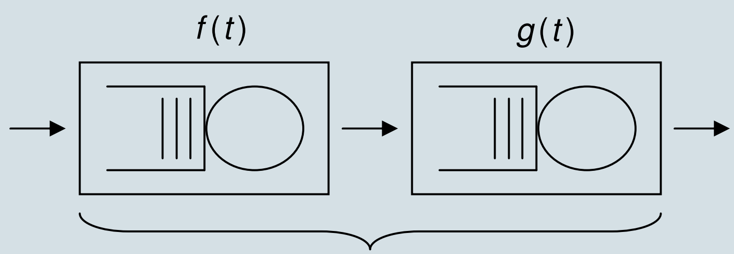


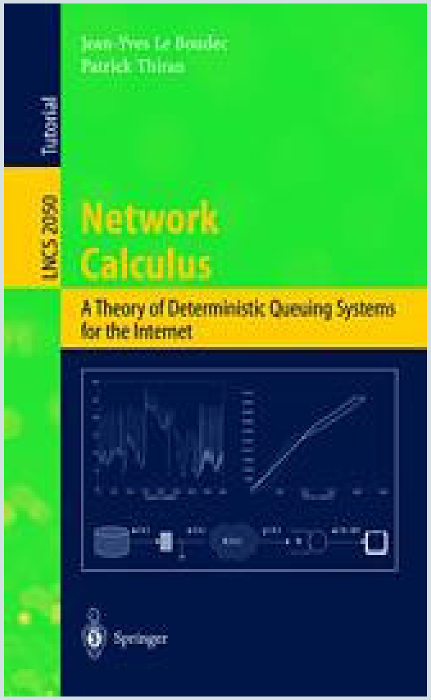
$$h(t) = (f \otimes g)(t) = \inf_u [f(t-u) + g(u)]$$

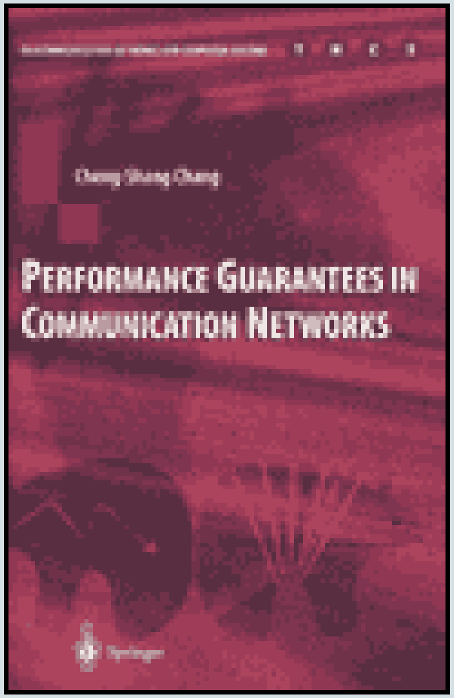
Conclusion

- Network Calculus is a set of tools and theories for the deterministic analysis of communication networks
- A new system theory, which applies min-plus algebra to communication networks
- Applicability needs to be evaluated carefully as still worst case performance is determined

Further Reading

ica1www.epfl.ch/PS_files/NetCal.htm





Cheng-Shang Chang,
Performance Guarantees in Communication Networks,

ISBN:1852332263

Legendre Transform

Eigenfunctions:

Convolution $$e^{j2\pi st} * g(t) = \int_{-\infty}^{+\infty} e^{j2\pi s(t-u)} g(u)du = e^{j2\pi st} \cdot G(s)$$

Min-Plus De-convolution $$(b+s\cdot t) \oslash g(t) = \sup_u \left[b+s\cdot(t+u)-g(u)\right] = (b+s\cdot t)+G(s)$$

Convex conjugate $$G(s) = \sup_u \left[s\cdot u - g(u)\right]$$

Concave conjugate $$H(s) = \inf_u \left[s\cdot u - h(u)\right]$$

(Convex conjugate and concave conjugate together: Fenchel Conjugates)

Fenchel Conjugate of a continuously differentiable function
is called Legendre Transform

Min-plus de-convolution

$$h(t) = f(t) \oslash g(t)$$

(transform pair)

$$H(s) = F(s) - G(s)$$

f concave,
g convex

Min-plus convolution

$$h(t) = f(t) \otimes g(t)$$

(transform pair)

$$H(s) = F(s) + G(s)$$

f convex,
g convex

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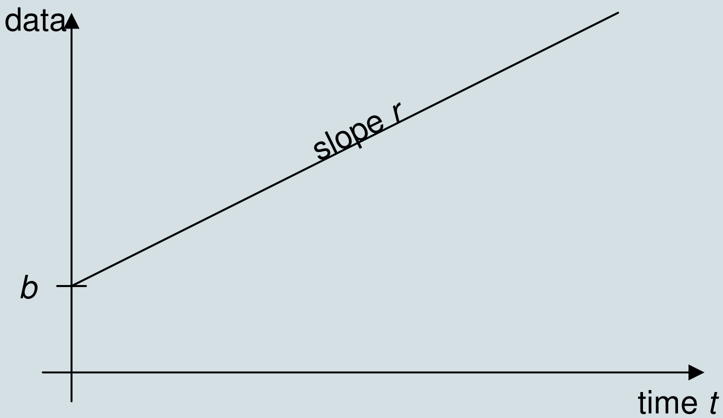
Conjugate Arrival Curves



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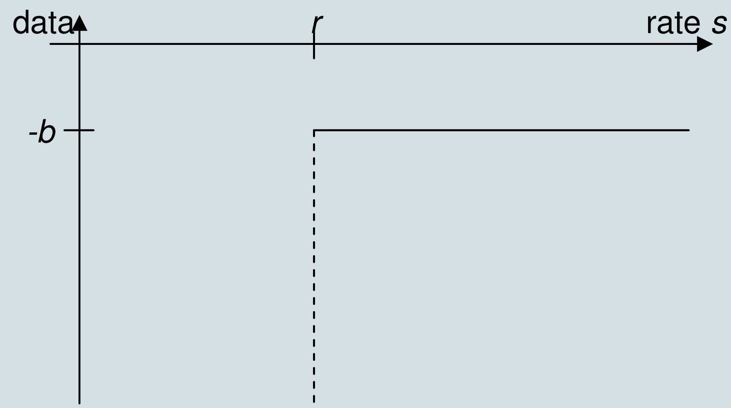
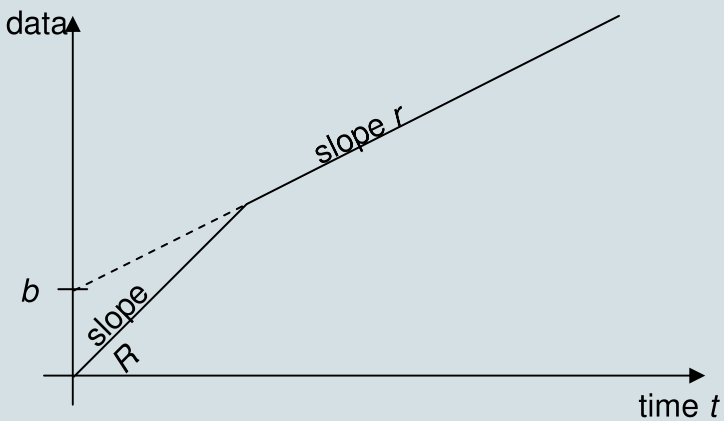


backlog bound at a constant rate server with rate *s*



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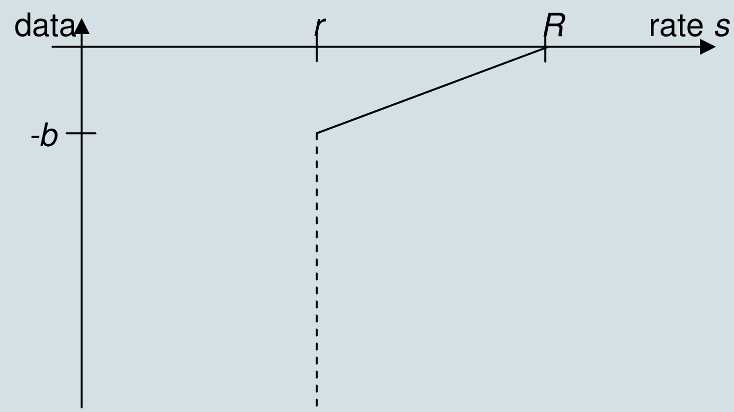

backlog bound at a constant rate server with rate *s*

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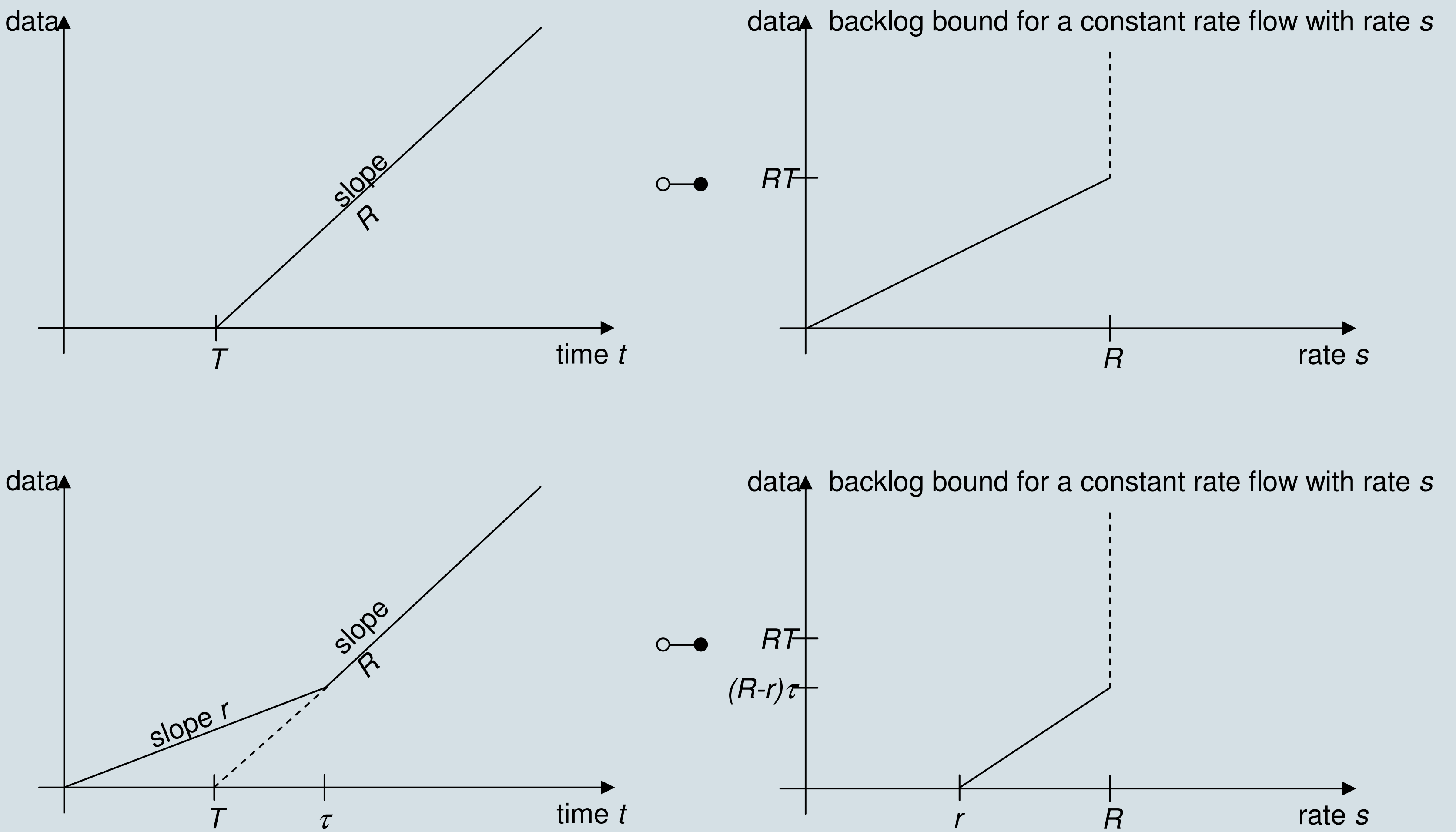
Conjugate Service Curves



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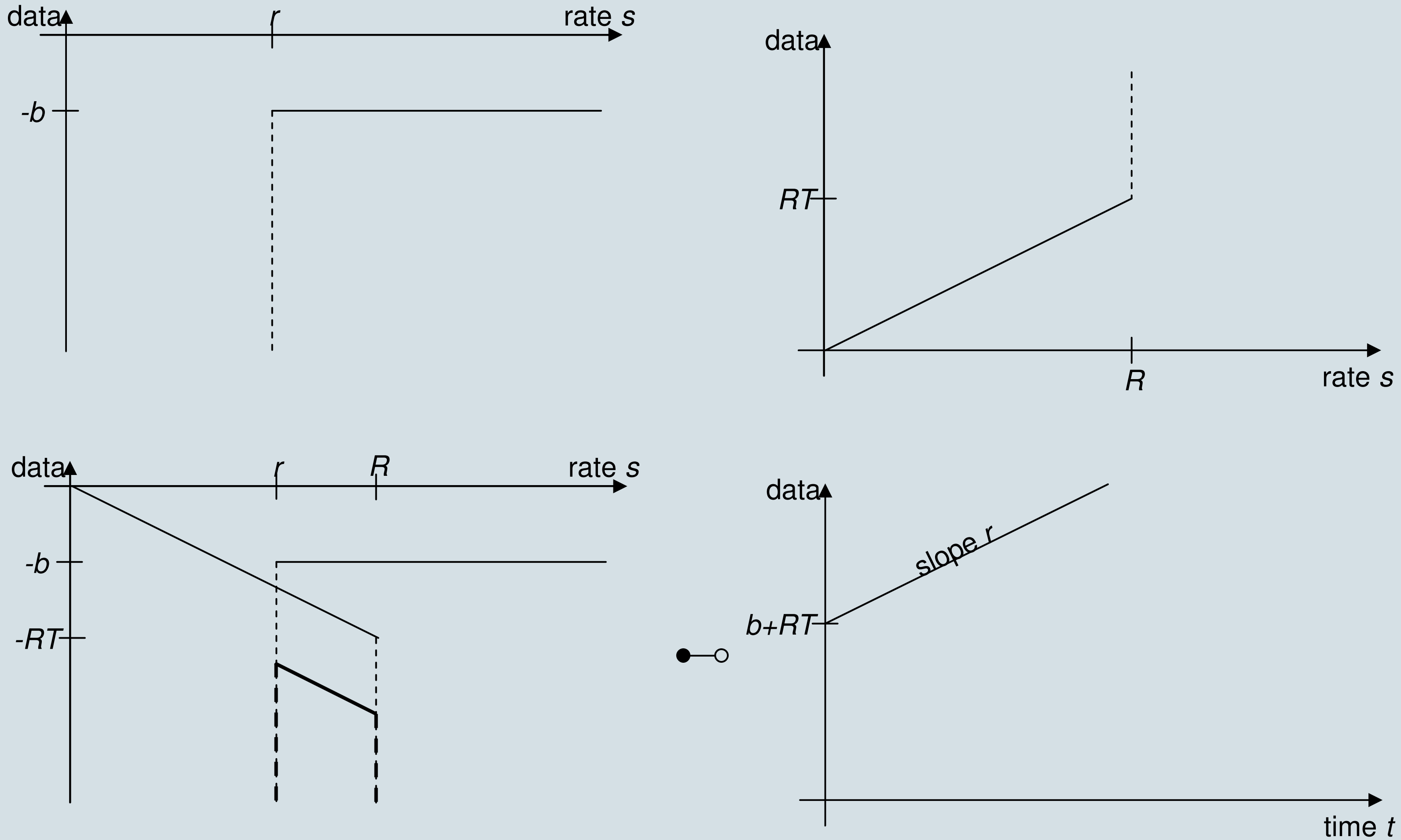
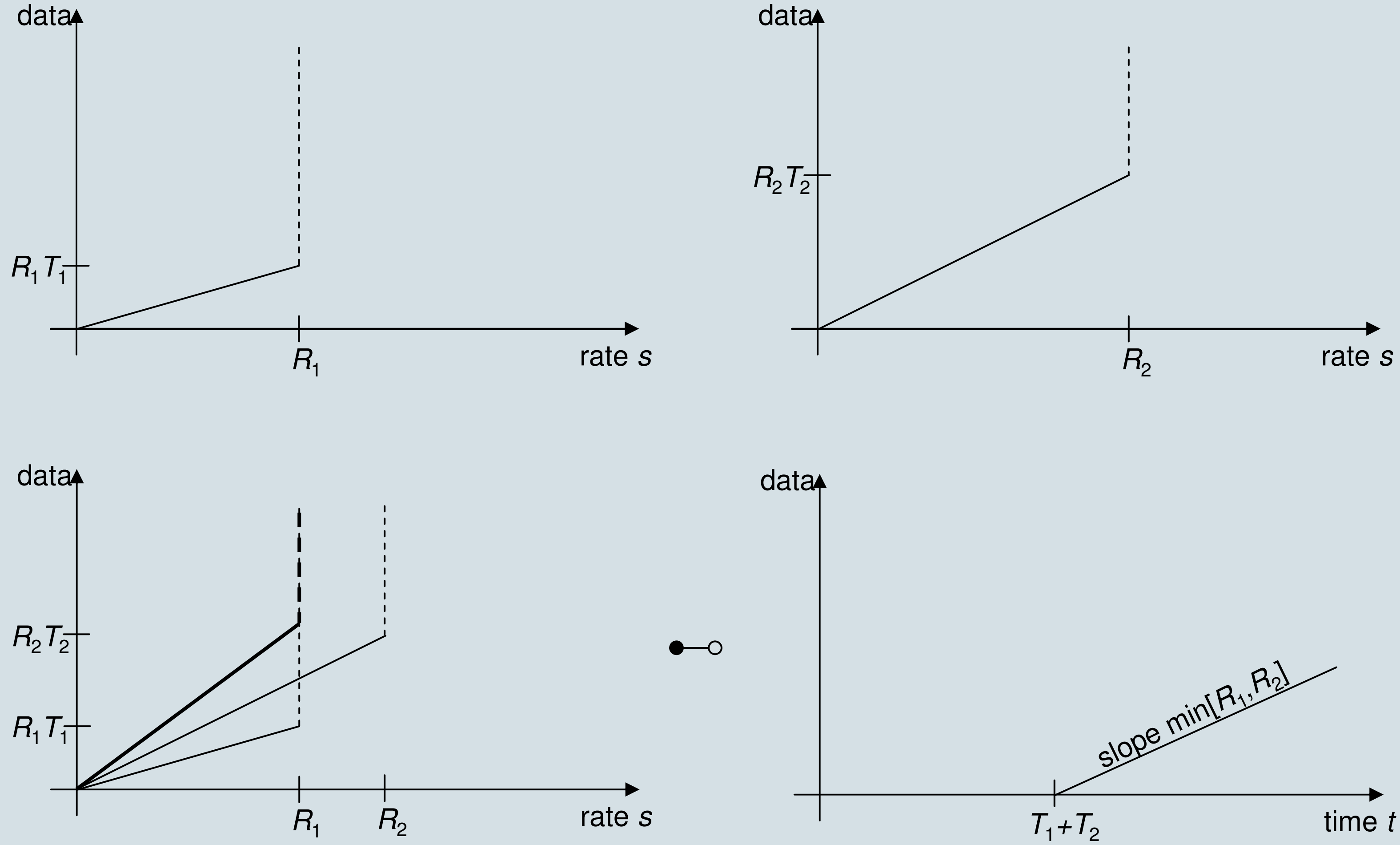
Server Output



Concatenation



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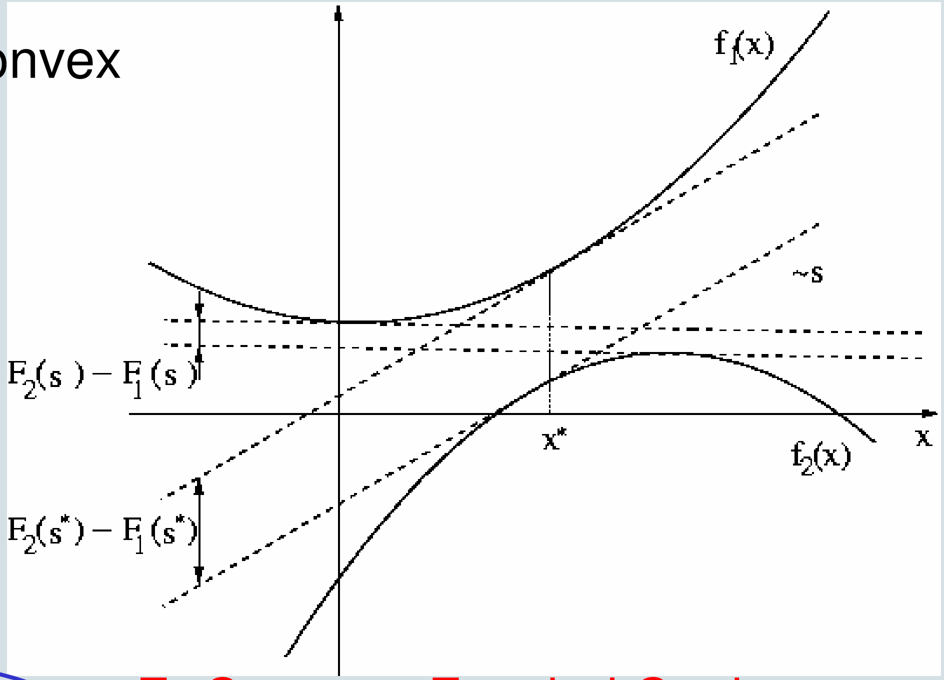
Fenchel Duality Theorem

$$\text{minimize} \quad f_1(x) - f_2(x)$$
$$\text{s.t.} \quad x \in \mathbb{R}$$

f_1,-f_2 convex

$$\text{minimize} \quad f_1(y) - f_2(z)$$
$$\text{s.t.} \quad y - z = 0$$

$$L(y,z,s) = f_1(y) - f_2(z) + s(y-z)$$



$$q(s) = \inf_{y,z \in \mathbb{R}} \{f_1(y) - f_2(z) + s(y-z)\}$$
$$= \inf_{z \in \mathbb{R}} \{sz - f_2(z)\} - \sup_{y \in \mathbb{R}} \{sy - f_1(y)\}$$

F_2 Concave Fenchel Conjugate

F_1 Convex Fenchel Conjugate

Duality Theory:

$$\text{Maximize} \quad q(s) = F_2(s) - F_1(s)$$
$$\text{s.t.} \quad s \in \mathbb{R}$$

Conjugate Performance Bounds

Conjugate Delay Bound:

$$\text{maximize} \quad d$$
$$\text{s.t.} \quad \alpha(t) = \beta(t+d)$$

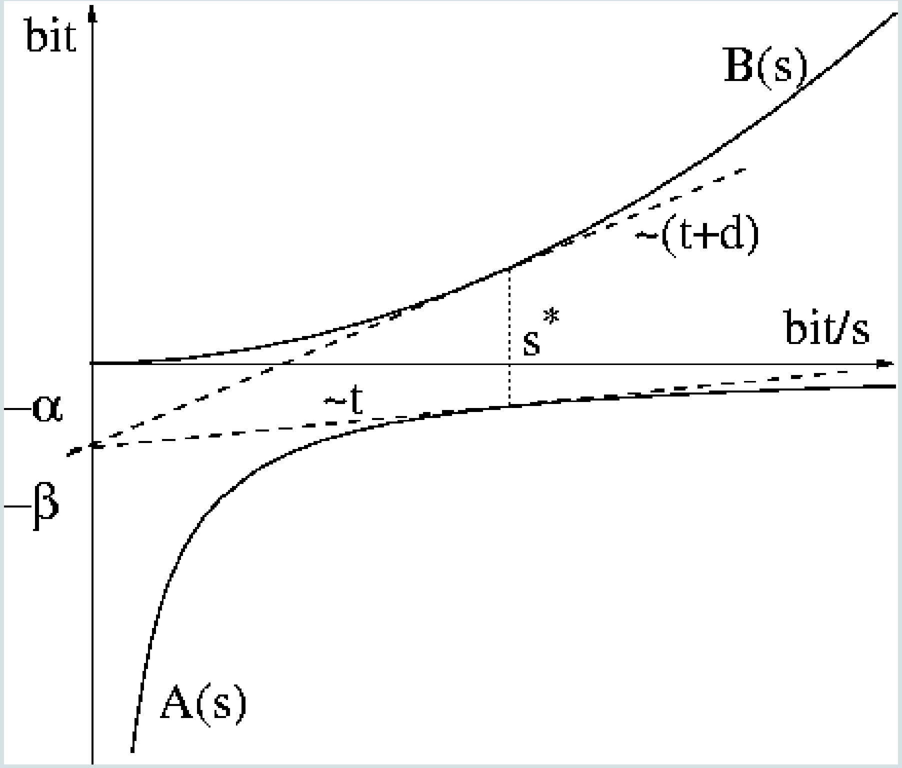
$$L(t,s) = d + s(\alpha(t) - \beta(t+d))$$

Biconjugates of closed functions

$$A(s) = \inf_{t \in \mathbb{R}} [st - \alpha(t)]$$
$$\alpha(t) = \inf_{s \in \mathbb{R}} [ts - A(s)]$$

$$\frac{\partial \alpha(t)}{\partial t} = \frac{\partial \beta(t+d)}{\partial t} \rightarrow \text{ tangents in one s}$$

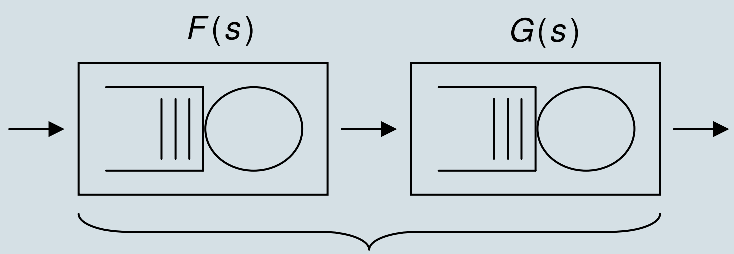
$$\frac{\partial L}{\partial s} \rightarrow \alpha(t) = \beta(t+d) \rightarrow \text{Identical Ordinates}$$



Essence of Conjugate Network Calculus

R.T. Rockafellar, Convex Analysis, Princeton University Press, 1972:

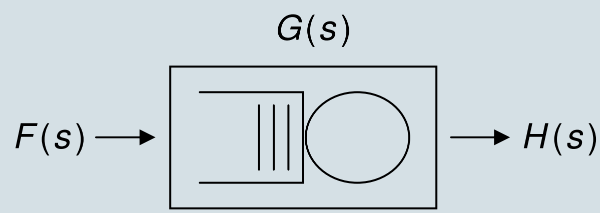
f convex, *g* convex



$$H(s) = F(s) + G(s) = \sup_t [st - f(t)] + \sup_t [st - g(t)]$$

M. Fidler, S. Recker, *A Dual Approach to Network Calculus Applying the Legendre Transform*, QoSIP 2005, Catania, Italy:

f concave, *g* convex



$$H(s) \leq F(s) - G(s) = \inf_t [st - f(t)] - \sup_t [st - g(t)]$$

M. Fidler, S. Recker, *Network Calculus and Conjugate Duality in Network Performance Analysis*, ITC19, Beijing, China:

Backlog Bound: $Q = -\sup_{s \in \mathbb{R}} [A(s) - B(s)]$

Delay Bound: Difference of slopes of tangents to A(s*) and B(s*), which have the same ordinate

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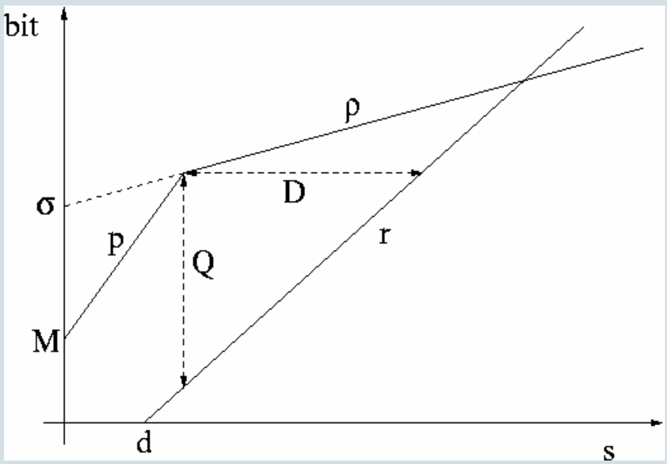
Token Bucket – Latency-Rate Example

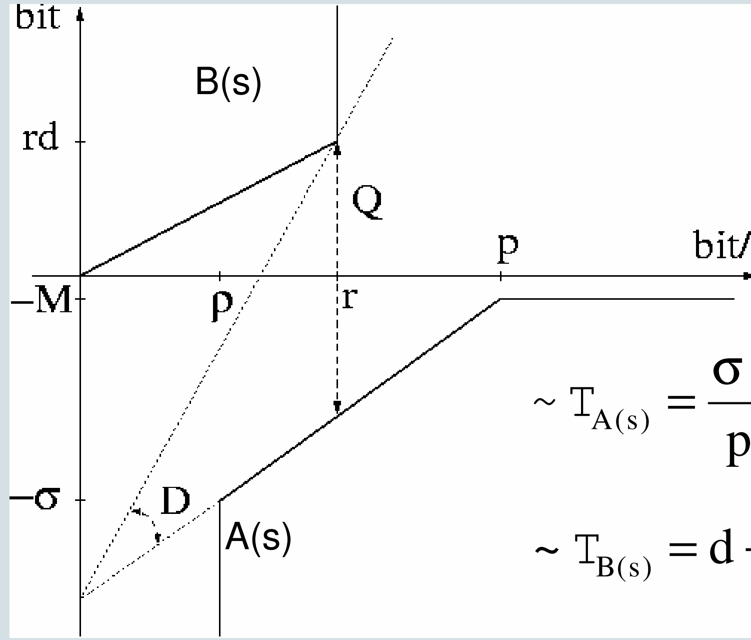
With $p \geq r \geq \rho$



$$\frac{\sigma - M}{p - \rho} > d :$$

$$Q = \sigma + \rho d + \left(\frac{\sigma - M}{p - \rho}\right)(\rho - r)$$

$$D = \frac{M}{r} + d + \frac{\sigma - M}{p - \rho}\frac{p - r}{r}$$



$$\sim T_{A(s)} = \frac{\sigma - M}{p - \rho}$$

$$\sim T_{B(s)} = d + \frac{\sigma}{\rho} + \frac{\sigma - M}{p - \rho}\frac{\rho}{r}$$

$$Q = \sigma + \rho d - \frac{\sigma - M}{p - \rho}(r - \rho)$$

$$D = d + \frac{\sigma}{\rho} + \frac{\sigma - M}{p - \rho}\left(\frac{\rho}{r} - 1\right)$$

$$= d + \frac{\sigma(p - r) - M(\rho - r)}{r(p - \rho)}$$