High-Level Loop Transformations and Polyhedral Compilation

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Outline

Loop Distribution

```
L: for (int i = 1; i < 100; ++i) {
        A[i] = f(i);
        B[i] = A[i] + A[i - 1];
}
```

Can this loop be parallelized?

Requirement:
writes of iteration do not conflict with reads/writes of other iteration

L[1]: W(A[1]) R(A[1]) R(A[0]) W(B[1])
L[2]: W(A[2]) R(A[2]) R(A[1]) W(B[2])

Loop distribution

```
L1: for (int i = 1; i < 100; ++i)
        A[i] = f(i);
L2: for (int i = 1; i < 100; ++i)
        B[i] = A[i] + A[i - 1];
```

No conflicts between iterations of L1 ⇒ can be run in parallel
No conflicts between iterations of L2 ⇒ can be run in parallel

Loop Distribution

```
L: for (int i = 1; i < 100; ++i) {
        A[i] = f(i);
        B[i] = A[i] + A[i + 1];
}
```

Can this loop be parallelized?

Requirement:
writes of iteration do not conflict with reads/writes of other iteration

L[1]: W(A[1]) R(A[1]) R(A[2]) W(B[1])
L[2]: W(A[2]) R(A[2]) R(A[3]) W(B[2])

Loop distribution changes meaning!

```
L1: for (int i = 1; i < 100; ++i)
        A[i] = f(i);
L2: for (int i = 1; i < 100; ++i)
        B[i] = A[i] + A[i + 1];
```

before distribution, L[1] reads A[2] value written before code fragment
after distribution, L2[1] reads A[2] value written by L1[2]

Loop Fusion

```
L1: for (int i = 0; i < 100; ++i)
        A[i] = f(i);
L2: for (int i = 0; i < 100; ++i)
        B[i] = g(A[i]);
```

Assume A does not fit in the cache
⇒ elements get evicted and reloaded for use in L2

Loop fusion (changes execution order ⇒ may not preserve meaning)

```
for (int i = 0; i < 100; ++i) {
        A[i] = f(i);
        B[i] = g(A[i]);
}
```

- ⇒ elements of A get reused immediately
- ⇒ better locality

Loop Transformations Loop Fusion May 30, 2017 6 / 82

Loop Fusion

```
L1: for (int i = 0; i < 100; ++i)
        A[i] = f(i);
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Loop fusion (changes execution order ⇒ may not preserve meaning)

```
for (int i = 0; i < 100; ++i) {
        A    = f(i);
        B[i] = g(A   );
}
```

- ⇒ elements of A get reused immediately
- ⇒ better locality

If A not needed outside code fragment

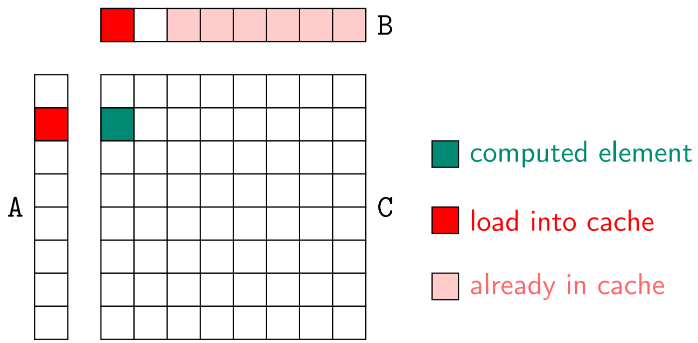
- ⇒ array can be replaced by a scalar
- ⇒ memory compaction

Loop Tiling

[17, 35]

```
L1: for (int i = 0; i < 8; ++i)
L2:     for (int j = 0; j < 8; ++j)
            C[i][j] = A[i] * B[j];
```

Assume B does not fit in the cache
⇒ elements get (re)loaded and evicted in every iteration of L1



Loop Transformations Loop Tiling May 30, 2017 7 / 82

Loop Tiling

[17, 35]

```
L1: for (int i = 0; i < 8; ++i)
L2:     for (int j = 0; j < 8; ++j)
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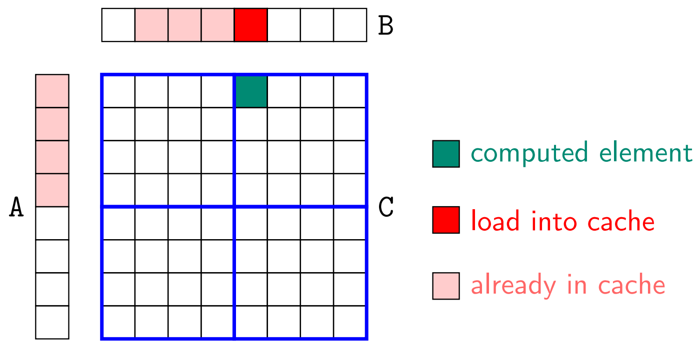


⇒ compute C in tiles, e.g., 4×4

Loop Tiling

[17, 35]

```
L1: for (int i = 0; i < 8; ++i)
L2:     for (int j = 0; j < 8; ++j)
                C[i][j] = A[i] * B[j];
```

Assume `B` does not fit in the cache
⇒ elements get (re)loaded and evicted in every iteration of `L1`

Loop tiling (changes execution order ⇒ may not preserve meaning)

```
for (int ti = 0; ti < 8; ti += 4)
    for (int tj = 0; tj < 8; tj += 4)
        for (int i = ti; i < ti + 4; ++i)
            for (int j = tj; j < tj + 4; ++j)
                C[i][j] = A[i] * B[j];
```


Motivation

- Computer architectures are becoming more difficult to program efficiently
  - multiple levels of parallelism
  - non-uniform memory architectures

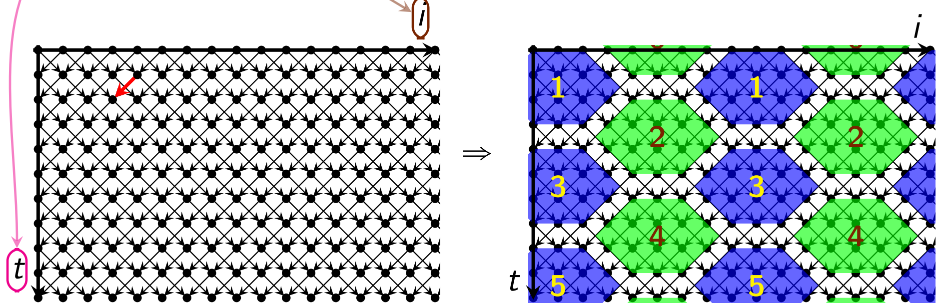
⇒ Advanced compiler optimizations are required
  - hierarchical partitioning and reordering of operations (e.g., parallelization, loop fusion, . . .)
  - mapping to different processing units
  - memory transfers between processing units

⇒ Global view of individual operations is required

⇒ Polyhedral Model

Polyhedral Compilation — Example

```
for (t = 0; t < T; t++)
  for (i = 1; i < N - 1; i++)
    A[(t+1)%2][i] = A[t%2][i-1] + A[t%2][i+1];
```



1. Extract polyhedral model
   ⇒ each dynamic instance represented by (t, i) pair
2. Compute dependences
   ⇒ iteration $t = 2, i = 3$ depends on iteration $t = 1, i = 4$
3. Compute schedule respecting dependences
   ⇒ tiles with same number can be executed in parallel
   ⇒ rows within tiles can be executed in parallel

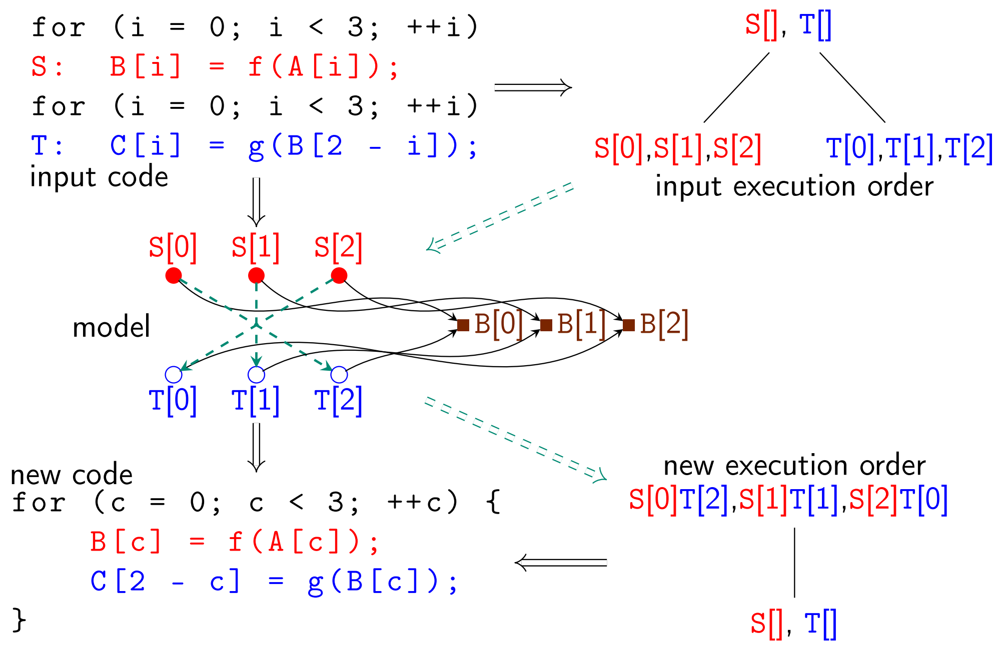
Polyhedral Model

[27]

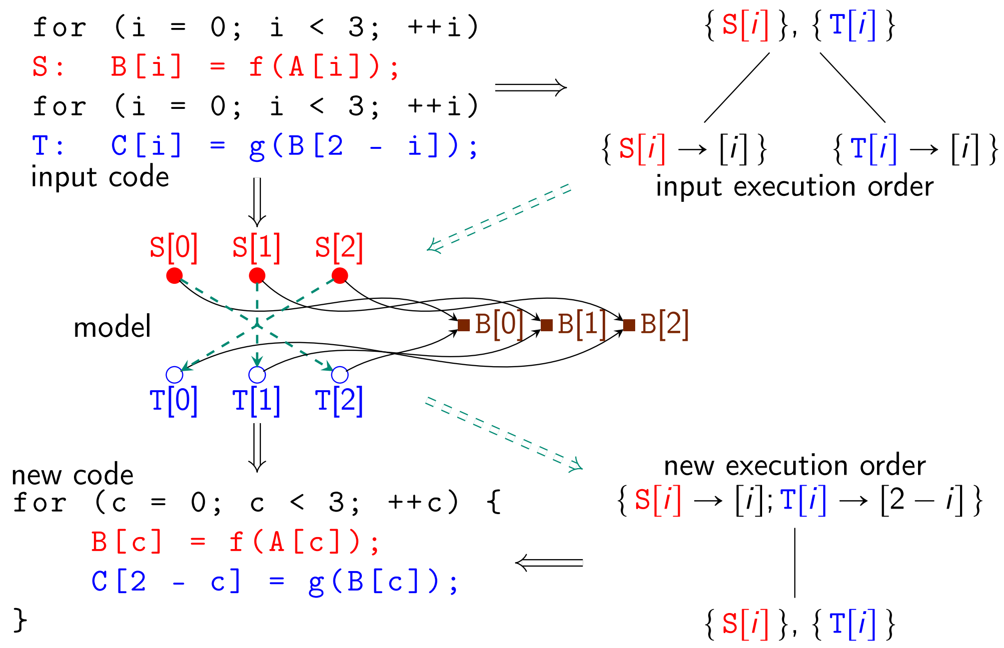
Key features
- instance based
  ⇒ statement *instances*
  ⇒ array *elements*
- compact representation based on polyhedra or similar objects
  ⇒ Presburger sets and relations
  ⇒ . . .

Main constituents of program representation
- Instance Set
  ⇒ the set of all statement instances
- Access Relations
  ⇒ the array elements accessed by a statement instance
- Dependences
  ⇒ the statement instances that depend on a statement instance
- Schedule
  ⇒ the relative execution order of statement instances
- Context
  ⇒ constraints on parameters

Polyhedral Model — Example



Polyhedral Model — Example



Polyhedral Model

[27]

Key features

- instance based
  - ⇒ statement *instances*
  - ⇒ array *elements*
- compact representation based on polyhedra or similar objects
  - ⇒ Presburger sets and relations defined by Presburger formula
  - ⇒ ...
- quasi-affine expression (no multiplication)
  - variable
  - constant integer number
  - constant symbol
  - addition (+), subtraction (−)
  - integer division by integer constant d ($\lfloor\cdot/d\rfloor$)
- Presburger formula
  - true
  - quasi-affine expression
  - less-than-or-equal relation ($\leqslant$)
  - equality ($=$)
  - first order logic connectives: $\wedge$, $\vee$, $\neg$, $\exists$, $\forall$

Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:     C[i][j] = 0;
        for (int k = 0; k < K; k++)
S2:         C[i][j] = C[i][j] + A[i][k] * B[k][j];
    }
```

- Instance Set (set of statement instances)

$$\{\, \mathtt{S1}[i,j] : 0 \leqslant i < M \wedge 0 \leqslant j < N;\ \mathtt{S2}[i,j,k] : 0 \leqslant i < M \wedge 0 \leqslant j < N \wedge 0 \leqslant k < K \,\}$$

- Access Relations (accessed array elements; W: write, R: read)

$$W = \{\, \mathtt{S1}[i,j] \to \mathtt{C}[i,j]; \mathtt{S2}[i,j,k] \to \mathtt{C}[i,j] \,\}$$

$$R = \{\, \mathtt{S2}[i,j,k] \to \mathtt{C}[i,j]; \mathtt{S2}[i,j,k] \to \mathtt{A}[i,k]; \mathtt{S2}[i,j,k] \to \mathtt{B}[k,j] \,\}$$

Schedule Representation

[30]

Schedule S keeps track of relative execution order of statement instances

$\Rightarrow$ for each pair of statement instances $\mathbf{i}$ and $\mathbf{j}$, schedule determines

- $\mathbf{i}$ executed before $\mathbf{j}$ ($\mathbf{i} <_S \mathbf{j}$),
- $\mathbf{i}$ executed after $\mathbf{j}$ ($\mathbf{j} <_S \mathbf{i}$), or
- $\mathbf{i}$ and $\mathbf{j}$ may be executed simultaneously

Schedule trees form a combined hierarchical schedule representation

- Main constructs:
  - affine schedule: instances are executed according to affine function
  - *sequence*: partitions instances through child *filters* executed in order
- Order of instances determined by outermost node that separates them
- Deriving schedule tree from AST
  - `for` loop $\Rightarrow$ affine schedule corresponding to loop iterator
  - compound statement $\Rightarrow$ sequence

Parametric Example: Matrix Multiplication

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for (int i = 0; i < M; i++)
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S1:     C[i][j] = 0;
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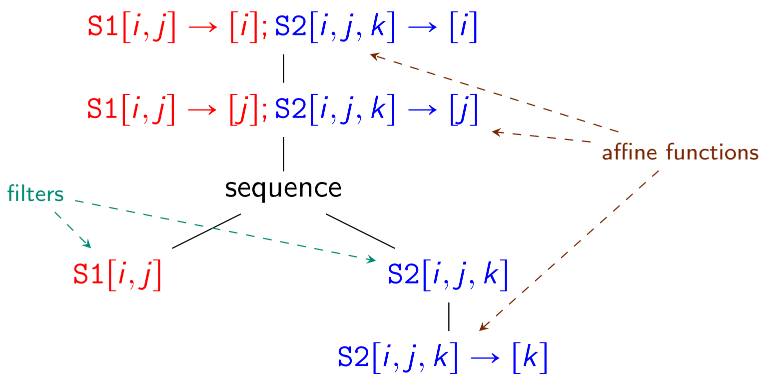
    }
```



Schedule Representation

[30]

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- $\mathbf{i}$ executed after $\mathbf{j}$ ($\mathbf{j} <_S \mathbf{i}$), or
- $\mathbf{i}$ and $\mathbf{j}$ may be executed simultaneously

Schedule trees form a combined hierarchical schedule representation

- Main constructs:
  - affine schedule: instances are executed according to affine function
  - *band*: nested sequence of affine functions called its *members*; combined multi-dimensional affine function is called the *partial schedule* of the band
  - *sequence*: partitions instances through child *filters* executed in order
- Order of instances determined by outermost node that separates them
- Deriving schedule tree from AST
  - `for` loop $\Rightarrow$ affine schedule corresponding to loop iterator
  - compound statement $\Rightarrow$ sequence

Named Presburger Relation Schedules

Schedule tree with single (band) node

Flattening a schedule tree

- two nested band nodes
  - $\Rightarrow$ replace by single band node with concatenated partial schedule
- sequence with as children either leaves or trees consisting of a single band node
  - $\Rightarrow$ treat leaves as zero-dimensional band nodes
  - $\Rightarrow$ pad lower-dimensional bands (e.g., with zero)
  - $\Rightarrow$ construct one-dimensional band assigning increasing values to children
  - $\Rightarrow$ combine one-dimensional band with children

Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:    C[i][j] = 0;
       for (int k = 0; k < K; k++)
S2:        C[i][j] = C[i][j] + A[i][k] * B[k][j];
    }
```

$$\texttt{S1}[i,j] \to [i,j,0,0]; \texttt{S2}[i,j,k] \to [i,j,1,k]$$

Loop Transformations and the Polyhedral Model

Loop transformations result in
different execution order of statement instances
⇒ different schedule

Polyhedral model can be used to

- evaluate a schedule and/or
- construct a schedule

Polyhedral schedules can represent (combinations of)

- loop distribution
- loop fusion
- loop tiling
- ...

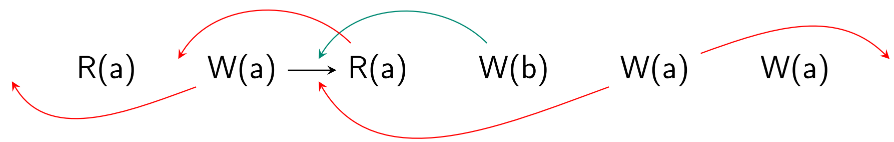
Schedule Properties

- Validity
  New schedule should preserve meaning
- Parallelism
  Can the iterations of a given loop be executed in parallel?
- Locality
  Statement instances scheduled closely to each other
- Tilability
  Can a given schedule band be tiled?

Schedule Validity

[3]

New schedule should preserve meaning



Internal restrictions

- No read of a value may be scheduled before the write of the value
- No other write to same memory location may be scheduled in between

External restrictions (on non-temporaries)

- No write may be scheduled before initial read from a memory location
- No write may be scheduled after last write to a memory location

Sufficient conditions:

- Every read of a memory location is scheduled after every preceding write to the same memory location
- Every write to a memory location is scheduled after every preceding read or write to the same memory location

Dependences

Sufficient conditions for validity of schedule S:

- Every read of a memory location is scheduled after every preceding write to the same memory location
- Every write to a memory location is scheduled after every preceding read or write to the same memory location

Dependence relation D: pairs of statement instances

- accessing the same memory location
- of which at least one is a write
- with the first executed before the second in original code

Sufficient condition:

$$\forall \mathbf{i} \to \mathbf{j} \in D : \mathbf{i} <_S \mathbf{j}$$

Dependence Analysis

Recall: sufficient conditions for validity of schedule S:

$$\forall \mathbf{i} \to \mathbf{j} \in D : \mathbf{i} <_S \mathbf{j}$$

Dependence relation D: pairs of statement instances

- accessing the same memory location
- of which at least one is a write
- with the first executed before the second in original code

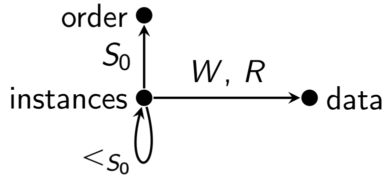
Computation:

$$D = ((W^{-1} \circ R) \cup (W^{-1} \circ W) \cup (R^{-1} \circ W)) \cap (<_{S_0})$$

W: write access relation

R: read access relation

S_0: original schedule



Local Validity

Schedule validity:

$$\forall \mathbf{i} \to \mathbf{j} \in D : \mathbf{i} <_S \mathbf{j}$$

Consider subset of *local* dependences L

At outermost node: $L = D$

Current node

- band node with partial schedule f

  $$\forall \mathbf{i} \to \mathbf{j} \in L : f(\mathbf{i}) \preccurlyeq_{lex} f(\mathbf{j})$$

  Carried dependences: $\mathbf{i} \to \mathbf{j} \in L : f(\mathbf{i}) \neq f(\mathbf{j})$
  $\Rightarrow$ no longer need to be considered in nested nodes
  Remaining dependences: $L' = \{\, \mathbf{i} \to \mathbf{j} \in L : f(\mathbf{i}) = f(\mathbf{j}) \,\}$
- sequence node with child position p and filters F_k

  $$\forall \mathbf{i} \to \mathbf{j} \in L : p(\mathbf{i}) \leqslant p(\mathbf{j})$$

  Carried dependences: $\mathbf{i} \to \mathbf{j} \in L : p(\mathbf{i}) \neq p(\mathbf{j})$
  Remaining dependences in child c: $L' = \{\, \mathbf{i} \to \mathbf{j} \in L : \mathbf{i}, \mathbf{j} \in F_c \,\}$
- leaf node: $L = \varnothing$

Loop Distribution Validity

```
for (int i = 1; i < 100; ++i) {
S:      A[i] = f(i);
T:      B[i] = A[i] + A[i - 1];
}
```

$\{\, \mathtt{S}[i] \to [i]; \mathtt{T}[i] \to [i] \,\}$

$\{\, \mathtt{S}[i] \,\}, \{\, \mathtt{T}[i] \,\}$

Dependences:

$$\{\, \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100; \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \,\}$$

$\{\, \mathtt{S}[i] \to [i]; \mathtt{T}[i] \to [i] \,\}$
satisfied: $\{\, \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100; \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \,\}$
carried: $\{\, \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \,\}$
$\{\, \mathtt{S}[i] \,\}, \{\, \mathtt{T}[i] \,\}$
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Loop Distribution Validity

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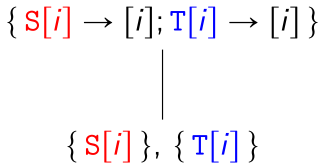

Dependences:

$$\{ \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100; \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \}$$

Loop distribution

```
for (int i = 1; i < 100; ++i)
          A[i] = f(i);
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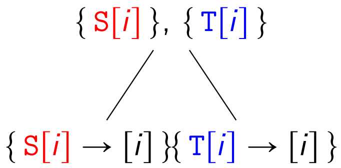

$\{ \mathtt{S}[i] \}, \{ \mathtt{T}[i] \}$
satisfied: $\{ \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100; \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \}$
carried: $\{ \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100; \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \}$

Loop Distribution Validity

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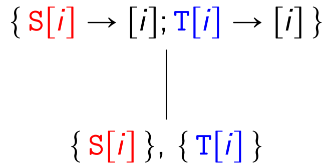

Dependences:

$$\{ \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100; \mathtt{T}[i] \to \mathtt{S}[i+1] : 1 \leqslant i, i+1 < 100 \}$$

Loop distribution

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for (int i = 1; i < 100; ++i)
          A[i] = f(i);
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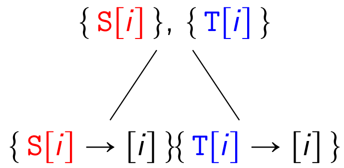

$\{ \mathtt{S}[i] \}, \{ \mathtt{T}[i] \}$
satisfied: $\{ \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100 \}$
violated: $\{ \mathtt{T}[i] \to \mathtt{S}[i+1] : 1 \leqslant i, i+1 < 100 \}$

Parallel Loops and Parallel Band Members

Recall:
Iterations of a given loop can be executed in parallel if
writes of iteration do not conflict with reads/writes of other iteration
iff there is no dependence between distinct iterations
(for any given iteration of the outer loops)

A band member with affine function f is parallel if

$$\forall \mathbf{i} \to \mathbf{j} \in L : f(\mathbf{i}) = f(\mathbf{j})$$

with L the local dependences

Loop Distribution and Parallelism

```
for (int i = 1; i < 100; ++i) {
S:        A[i] = f(i);
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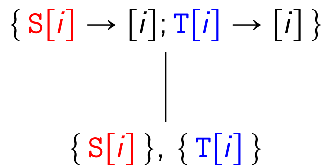

Dependences:

$$\{ \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100; \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \}$$

$\{ \mathtt{S}[i] \to [i]; \mathtt{T}[i] \to [i] \}$
local: $\{ \mathtt{S}[i] \to \mathtt{T}[i] : 1 \leqslant i < 100; \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \}$
conflict: $\{ \mathtt{S}[i] \to \mathtt{T}[i+1] : 1 \leqslant i, i+1 < 100 \}$
$\Rightarrow$ not parallel

Loop Distribution and Parallelism

```
for (int i = 1; i < 100; ++i) {
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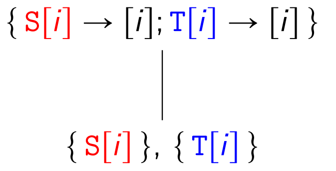

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Loop distribution

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          A[i] = f(i);
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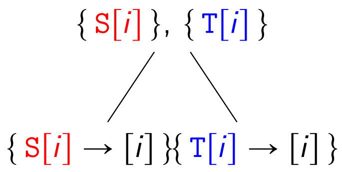


$\{ \mathtt{S}[i] \to [i] \}$
local: $\varnothing$
conflict: $\varnothing$
⇒ parallel

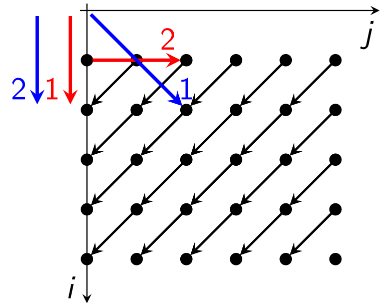
$\{ \mathtt{T}[i] \to [i] \}$
local: $\varnothing$
conflict: $\varnothing$
⇒ parallel

Parallelism Example

```
for (int i = 1; i < 6; ++i)
    for (int j = 0; j < 6; ++j)
S:        A[i][j] = f(A[i - 1][[j + 1]);
```

Dependences:

$$\{ \mathtt{S}[i,j] \to \mathtt{S}[i+1, j-1] : 1 \leqslant i, i+1 < 6 \wedge 0 \leqslant j, j-1 < 6 \}$$



original schedule:
$\mathtt{S}[i,j] \to [i,j]$
new schedule:
$\mathtt{S}[i,j] \to [i+j, i]$
$(i+j)$-direction is outer parallel

Decomposition: loop skewing + loop interchange

$[i,j] \to [i, i+j] \to [i+j, i]$

Locality

Statement instances **i** and **j** that reuse memory
⇒ scheduled closely to each other: $f(\mathbf{j}) - f(\mathbf{i})$ small

Types of locality:

- temporal locality
  ⇒ instances that access the same memory element
- spatial locality
  ⇒ instances that access adjacent memory elements

Sometimes further distinction made:

- self locality
  ⇒ pair of instances from same statement
- group locality
  ⇒ any pair of statement instances

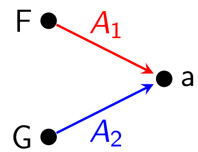
Temporal locality often restricted to
pairs of writes and reads that refer to the same value
⇒ dataflow

Array Dataflow Analysis

[14]

Given a read from an array element, what was the last write to the same array element before the read?

```
for (i = 0; i < N; ++i)
    for (j = 0; j < N - i; ++j)
F:        a[i+j] = f(a[i+j]);
for (i = 0; i < N; ++i)
G:  g(a[i]);
```



Access relations:
$A_1 = \{ \mathtt{F}[i,j] \to \mathrm{a}[i+j] : 0 \leqslant i < N \wedge 0 \leqslant j < N - i \}$
$A_2 = \{ \mathtt{G}[i] \to \mathrm{a}[i] : 0 \leqslant i < N \}$
Map to all writes: $R'' = A_1{}^{-1} \circ A_2 = \{ \mathtt{G}[i] \to \mathtt{F}[i', i - i'] : 0 \leqslant i' \leqslant i < N \}$
Map to all preceding writes:
$R' = R'' \cap (<_S)^{-1} = \{ \mathtt{G}[i] \to \mathtt{F}[i', i - i'] : 0 \leqslant i' \leqslant i < N \}$
Last preceding write: $R = \max_{<_S} R' = \{ \mathtt{G}[i] \to \mathtt{F}[i, 0] : 0 \leqslant i < N \}$

Tiling a Band

Input:

- band of affine schedule functions

$$f_1, f_2, \ldots, f_n$$

- tile sizes

$$T_1, T_2, \ldots, T_n$$

Steps (conceptually)

1. divide each direction into chunks of size T_i (strip-mining)

$$\lfloor f_1/T_1 \rfloor, f_1, \lfloor f_2/T_2 \rfloor, f_2, \ldots, \lfloor f_n/T_n \rfloor, f_n$$

   does not change execution order ⇒ always valid
2. combine the chunking (interchange)

$$\lfloor f_1/T_1 \rfloor, \lfloor f_2/T_2 \rfloor, \ldots, \lfloor f_n/T_n \rfloor, f_1, f_2, \ldots, f_n$$

   sufficient condition for interchange:
   all members are valid for local dependences at (top of) band
   ⇒ permutable band

Loop Tiling Example

```
for (int i = 0; i < 8; ++i)
    for (int j = 0; j < 8; ++j)
S:      C[i][j] = A[i] * B[j];
```

1. strip-mine

$$S[i,j] \to 4\lfloor i/4 \rfloor$$
$$S[i,j] \to i$$
$$S[i,j] \to 4\lfloor j/4 \rfloor$$
$$S[i,j] \to j$$

```
for (int ti = 0; ti < 8; ti += 4)
    for (int i = ti; i < ti + 4; ++i)
        for (int tj = 0; tj < 8; tj += 4)
            for (int j = tj; j < tj + 4; ++j)
                C[i][j] = A[i] * B[j];
```


Loop Tiling Example

```
for (int i = 0; i < 8; ++i)
    for (int j = 0; j < 8; ++j)
S:      C[i][j] = A[i] * B[j];
```

1. strip-mine
2. interchange

$$S[i,j] \to 4\lfloor i/4 \rfloor$$
$$S[i,j] \to 4\lfloor j/4 \rfloor$$
$$S[i,j] \to i$$
$$S[i,j] \to j$$

```
for (int ti = 0; ti < 8; ti += 4)
    for (int tj = 0; tj < 8; tj += 4)
        for (int i = ti; i < ti + 4; ++i)
            for (int j = tj; j < tj + 4; ++j)
                C[i][j] = A[i] * B[j];
```


Operations on Polyhedral Model

- Model Extraction
  - Input: AST
  - Output: instance set, access relations, original schedule
- Dependence analysis
  - Input: instance set, access relations, original schedule
  - Output: dependence relations
- Scheduling
  - Input: instance set, dependence relations
  - Output: schedule
- AST generation (polyhedral scanning, code generation)
  - Input: instance set, schedule
  - Output: AST
- Data layout transformations
  - Input: access relations, dependence relations
  - Output: transformed access relations

Polyhedral Model Requirements

Requirements for basic polyhedral model: "regular" code

- Static control
  ⇒ control does not depend on input data
- Affine
  ⇒ all relevant expressions are (quasi-)affine
- No Aliasing
  ⇒ essentially no pointer manipulations

Note:

- polyhedral model may be *approximation* of input that does not strictly satisfy all requirements
- many *extensions* are available

Aliasing

[1]

Some possible ways of handling aliasing:

- use an input language that does not permit aliasing
- pretend the problem does not exist
- require user to ensure absence of aliasing
  ⇒ e.g., use `restrict` keyword
- handle as may-write
  ⇒ may lead to too many dependences
- check aliasing at run-time
  ⇒ use original code in case of aliasing

Polyhedral Scheduling

[10, 15]

Polyhedral model can be used to

- evaluate a schedule and/or
- construct a schedule

Some popular polyhedral schedulers:

- Feautrier
  - maximal inner parallelism
    ⇒ carry as many dependences as possible at outer bands
- Pluto
  - tilable bands
  - locality: $f(\mathbf{j}) - f(\mathbf{i})$ small
    ⇒ parallelism as extreme case: $f(\mathbf{j}) - f(\mathbf{i}) = 0$

Many other scheduling algorithms have been proposed

Data layout transformations

[12, 13]

- Memory compaction
  Reuse memory locations to store different data
  - ⇒ apply non-injective mapping to array elements
  - ⇒ reduce memory requirements
  - ⇒ extreme case: replace array by scalar

```
for (int i = 0; i < 100; ++i) {
        A[i] = f(i);
        B[i] = g(A[i]);
}
```

- Expansion
  Use different memory locations to store different data
  - ⇒ map different accesses to memory element to distinct locations
  - ⇒ increase scheduling freedom (e.g., more parallelism)

False Dependences

```
for (int i = 0; i < n; ++i) {
S:      t = f1(A[i]);
T:      B[i] = f2(t);
}
```

Dependences

- read-after-write ("true"): $\{ S[i] \to T[i'] : i' \geqslant i \}$
- write-after-read ("anti"): $\{ T[i] \to S[i'] : i' > i \}$ } "false"
- write-after-write ("output"): $\{ S[i] \to S[i'] : i' > i \}$ } "false"

False dependences not from dataflow, but from reuse of memory location t

Possible solution: expansion/privatization

```
for (int i = 0; i < n; ++i) {
S:      t[i] = f1(A[i]);
T:      B[i] = f2(t[i]);
}
```

- dataflow (subset of "true" dependences): $\{ S[i] \to T[i] \}$

Expansion

Assume:

- instance sets and access relations are static and exact
  ⇒ each read has exactly one corresponding write
- single read and write per statement
  ⇒ expanded array indexed by statement instance of write

```
for (int i = 0; i < n; ++i) {
S:      t = f1(A[i]);
T:      B[i] = f2(t);
}
```

Dataflow: $\{ S[i] \to T[i] \}$

```
for (int i = 0; i < n; ++i) {
S:      S[i] = f1(A[i]);
T:      B[i] = f2(S[i]);
}
```

⇒ only remaining dependences are dataflow induced

Maximal Static Expansion

[5]

```
for (int i = 0; i < n; ++i) {
S1:     t = f1(i);                t1[i] = f1(i);
S2:     A[i] = t;                 A[i] = t1[i];
S3:     t = f2(i);                t2[i] = f2(i);
S4:     if (f3(i))                if (f3(i))
S5:             t = f4(i);                t2[i] = f4(i);
S6:     B[i] = t;                 B[i] = t2[i];
}
```

Dataflow cannot be determined independently of run-time information

⇒ approximate dataflow
$\{ S1[i] \to S2[i]; S3[i] \to S6[i]; S5[i] \to S6[i] \}$

⇒ a read may be associated to more than one write

⇒ corresponding equivalence classes should not be expanded apart

Approximate Dataflow Analysis

How to compute dataflow in presence of data dependent control?

Two approaches

- Direct computation
  - distinguish between may- and must-writes
- Derived from exact run-time dependent dataflow
  - compute exact dataflow in terms of run-time information
  - exploit properties of run-time information
  - project out run-time information

May Writes

Keep track of whether write is possible or definite

- Must-writes
  Array elements are definitely written by statement instance
- May-writes
  Array elements are possibly written by statement instance
  - statement instance not necessarily executed

```
for (i = 0; i < n; ++i)
     if (A[i] > 0)
S:        B[i] = A[i];
```

    May-write: $\{ \texttt{S}[i] \to \texttt{B}[i] \}$
  - array element not necessarily accessed

```
int A[N];
/* ... */
T:  A[B[0]] = 5;
```

    May-write: $\{ \texttt{T}[] \to \texttt{A}[a] : 0 \leqslant a < N \}$

Must-write access relation is subset of may-write access relation

Approximate Dataflow — Direct Computation

- Read-after-write dependences
  - write and read access same memory location
  - write executed before the read

  ⇒ Approximate dataflow analysis with no must-writes
- Dataflow dependences
  - write and read access same memory location
  - write executed before the read
  - no intermediate write to same memory location
    ⇒ intermediate write kills dependence
- Approximate dataflow dependences
  - may-write and read access same memory location
  - may-write executed before the read
  - no intermediate must-write to same memory location
    ⇒ intermediate must-write kills dependence

Approximate Dataflow Analysis

How to compute dataflow in presence of data dependent control?

Two approaches

- Direct computation
  - distinguish between may- and must-writes
- Derived from exact run-time dependent dataflow
  - compute exact dataflow in terms of run-time information
  - exploit properties of run-time information
  - project out run-time information

Run-time Dependent Dataflow Analysis

[6, 32]

Approaches

- "fuzzy array dataflow analysis"
- "on-demand-parametric array dataflow analysis"

```
for (int i = 0; i < n; ++i) {
S1:     t = f1(i);
S2:     A[i] = t;
S3:     t = f2(i);
S4:     if (f3(i))
S5:             t = f4(i);
S6:     B[i] = t;
}
```

- Run-time dependent dataflow
  $\{ \texttt{S1}[i] \to \texttt{S2}[i]; \texttt{S3}[i] \to \texttt{S6}[i] : \beta_{\texttt{S6}}^{\texttt{S5}} = 0; \texttt{S5}[i] \to \texttt{S6}[i] : \beta_{\texttt{S6}}^{\texttt{S5}} = 1 \}$
  β_C^P: any potential source instance P is executed for sink C
  $\boldsymbol{\lambda}_C^P$: last potential source instance P executed for sink C
- Approximate dataflow (project out β and $\boldsymbol{\lambda}$)
  $\{ \texttt{S1}[i] \to \texttt{S2}[i]; \texttt{S3}[i] \to \texttt{S6}[i]; \texttt{S5}[i] \to \texttt{S6}[i] \}$

Representing Dynamic Conditions

```
N1: n = f();
    for (int k = 0; k < 100; ++k) {
M:      m = g();
        for (int i = 0; i < m; ++i)
            for (int j = 0; j < n; ++j)
A:              a[j][i] = g();
N2:     n = f();
    }
```

What is instance set (restricted to A statement)?
$\{ \texttt{A}[k,i,j] : 0 \leqslant k < 100 \wedge 0 \leqslant i < \texttt{m} \wedge 0 \leqslant j < \texttt{n} \}$?

⇒ no, m and n cannot be treated as symbolic constants
(they are modified inside k-loop)

$\{ \texttt{A}[k,i,j] : 0 \leqslant k < 100 \wedge 0 \leqslant i < \texttt{valueOf\_m}(k) \wedge 0 \leqslant j < \texttt{valueOf\_n}(k) \}$?

⇒ requires uninterpreted functions (of arity > 0)

Alternative: use overapproximation of instance set and keep track of which elements are executed

Representing Dynamic Conditions

```
N1: n = f();
    for (int k = 0; k < 100; ++k) {
M:      m = g();
        for (int i = 0; i < m; ++i)
            for (int j = 0; j < n; ++j)
A:              a[j][i] = g();
N2:     n = f();
    }
```

- Instance set: $\{ \texttt{A}[k,i,j] : 0 \leqslant k < 100 \wedge 0 \leqslant i \wedge 0 \leqslant j \}$
- Filter:
  - Filter access relations: reader → [writer → array element]
    - $F_1^{\texttt{A}} = \{ \texttt{A}[k,i,j] \to [\texttt{M}[k] \to \texttt{m}[]] \}$
    - $F_2^{\texttt{A}} = \{ \texttt{A}[0,i,j] \to [\texttt{N1}[] \to \texttt{n}[]]; \texttt{A}[k,i,j] \to [\texttt{N2}[k-1] \to \texttt{n}[]] : k \geqslant 1 \}$
  - Filter value relation:
    $V^{\texttt{A}} = \{ \texttt{A}[k,i,j] \to [m,n] : 0 \leqslant k \leqslant 99 \wedge 0 \leqslant i < m \wedge 0 \leqslant j < n \}$

Statement instance is executed iff values written by corresponding write accesses (through filter access relations) satisfy filter value relation

Parametric Array Dataflow Analysis

```
while (1) {
N:  n = f();
    a = g();
    if (n < 100)
H:      a = h();
    if (n > 200)
T:      t(a);
}
```

potential source (H: a); sink (T: a)

$I = \{ \texttt{H}[i] : i \geqslant 0; \texttt{T}[i] : i \geqslant 0 \}$
$F^{\texttt{H}} = \{ \texttt{H}[i] \to [\texttt{N}[i] \to \texttt{n}[]] \}$
$V^{\texttt{H}} = \{ \texttt{H}[i] \to [n] : i \geqslant 0 \wedge n < 100 \}$
$F^{\texttt{T}} = \{ \texttt{T}[i] \to [\texttt{N}[i] \to \texttt{n}[]] \}$
$V^{\texttt{T}} = \{ \texttt{T}[i] \to [n] : i \geqslant 0 \wedge n > 200 \}$

Is there any dataflow between potential source and sink at inner level?

- $M = \{ \texttt{T}[i] \to \texttt{H}[i] \}$
- $F^{\texttt{H}} \circ M \subseteq F^{\texttt{T}}$
  - ⇒ filter elements accessed by any potential source instance associated to sink instance forms subset of filter elements accessed by sink instance
  - ⇒ constraints on filter values at sink also apply at corresponding potential source: $V^{\texttt{T}} \circ M^{-1} = \{ \texttt{H}[i] \to [n] : i \geqslant 0 \wedge n > 200 \}$
- $(V^{\texttt{T}} \circ M^{-1}) \cap V^{\texttt{H}} = \emptyset$
  - ⇒ there can be no dataflow at inner level

Polyhedral Process Networks

[24]

- Main purpose: extract task level parallelism from dataflow graph

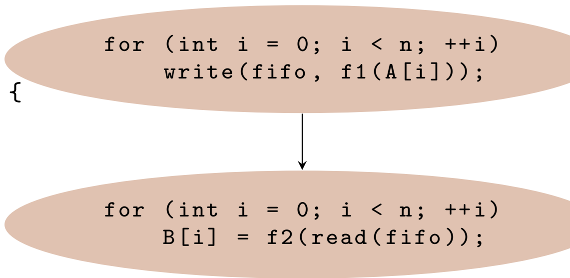
statement	→	process
flow dependence	→	communication channel

⇒ requires dataflow analysis

- Processes are mapped to parallel hardware (e.g., FPGA)

Example:

```
for (int i = 0; i < n; ++i) {
S:      t = f1(A[i]);
T:      B[i] = f2(t);
}
```



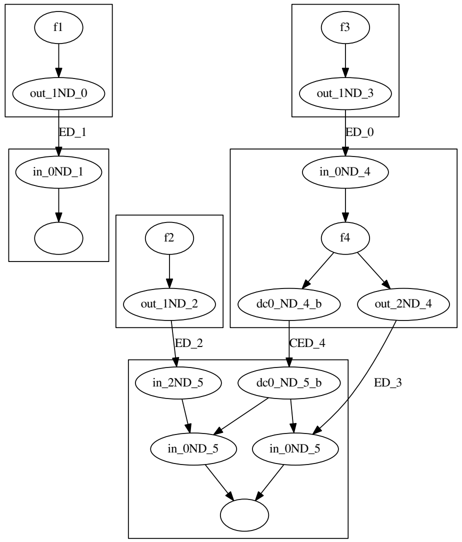
Process Networks with Dynamic Control

```
for (int i = 0; i < n; ++i) {
S1:     t = f1(i);
S2:     A[i] = t;
S3:     t = f2(i);
S4:     if (f3(i))
S5:             t = f4(i);
S6:     B[i] = t;
}
```

Run-time dependent dataflow:

$\{ \mathtt{S1}[i] \to \mathtt{S2}[i]; \mathtt{S3}[i] \to \mathtt{S6}[i] : \beta^{\mathtt{S5}}_{\mathtt{S6}} = 0;$

$\mathtt{S5}[i] \to \mathtt{S6}[i] : \beta^{\mathtt{S5}}_{\mathtt{S6}} = 1; \mathtt{S4}[i] \to \mathtt{S5}[i] \}$



Polyhedral Software

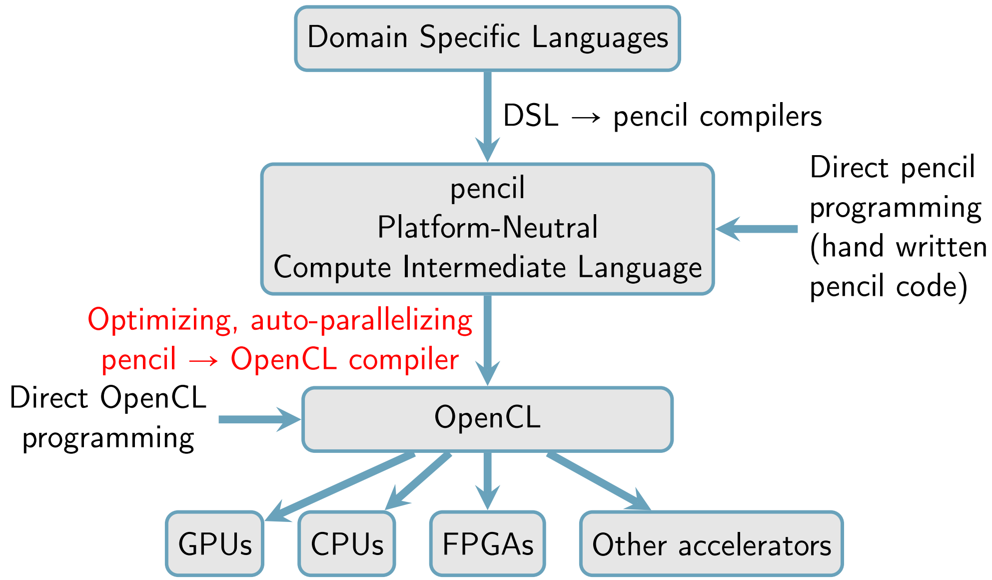
[4, 7, 8, 9, 10, 11, 16, 18, 19, 20, 21, 22, 23, 29, 31, 34]

http://polyhedral.info/software.html

- Core set manipulation libraries
  - integer sets: isl, omega(+), ...
  - rational sets: PolyLib, PPL, ...
- Model extraction
  - clan, pet, ...
- Dependence analysis
  - petit, candl, isl, FADA, ...
- Scheduler libraries
  - LetSee, isl, ...
- AST generation
  - omega(+), CLooG, isl, ...
- Source-to-source polyhedral compilers
  - Pluto, PoCC, PPCG, ...
- Compilers using polyhedral compilation
  - gcc/graphite, LLVM/Polly, ...

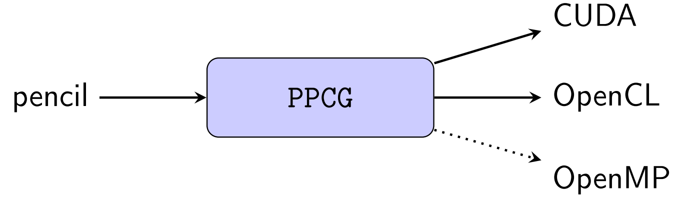
CARP Project (2011–2015)

Design tools and techniques to aid
Correct and Efficient Accelerator Programming



PPCG Overview

[31]



PPCG:

- detect/expose parallelism
- map parts of the code to an accelerator
- copy data to/from device
- introduce local copies of data

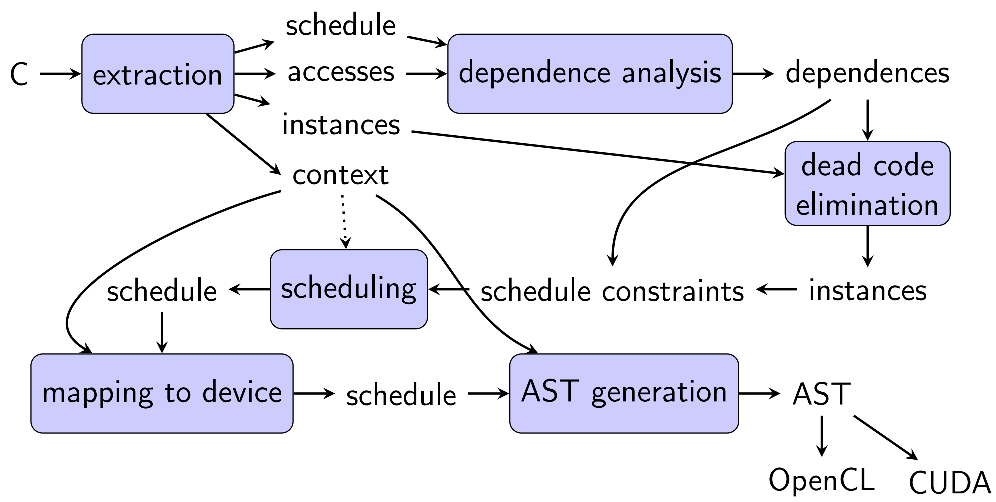
pencil:

- C99 with restrictions and some extra builtins and pragmas

pencil

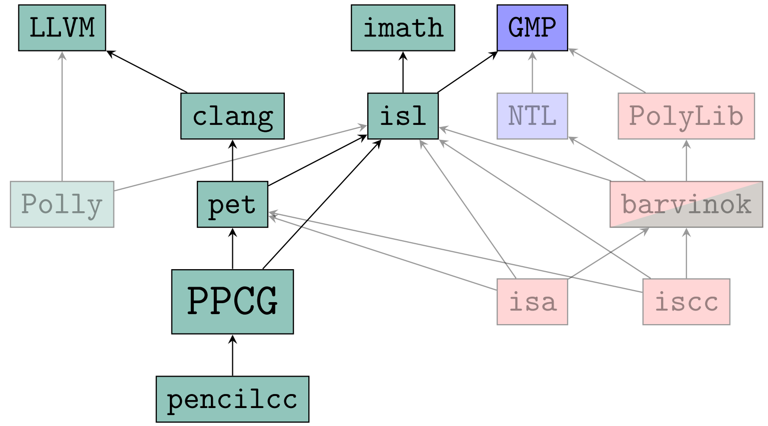
PPCG Internal Structure

[31]



Note: as currently implemented (version 0.07), not necessarily how it should be implemented

Connection with other Libraries and Tools



Licenses:
BSD/MIT
LGPL
GPL

`isl`: manipulates parametric affine sets and relations
`pet`: extracts polyhedral model from clang AST
`PPCG`: Polyhedral Parallel Code Generator
`pencilcc`: pencil compiler

Instance Set

Region that needs to be extracted may be

- marked by

```
#pragma scop
#pragma endscop
```

- autodetected (`--pet-autodetect`)

Internal structured dynamic control is encapsulated

```
for (int x = 0; x < n; ++x) {
A:      s = f();
B:      while (P(x, s))
                s = g(s);
C:      h(s);
}
```

Instance set: $\{ \mathtt{A}[x] : 0 \leqslant x < n; \mathtt{B}[x] : 0 \leqslant x < n; \mathtt{C}[x] : 0 \leqslant x < n \}$

Note: currently, internal order of accesses is lost
$\Rightarrow$ possible loss of accuracy in dependence analysis

Inlining

Enabled through C99 `inline` keyword on function definition

```
inline void set_diagonal(int n,
        float A[const restrict static n][n], float v)
{
        for (int i = 0; i < n; ++i)
U:              A[i][i] = v;
}

void f(int n, float A[const restrict static n][n])
{
#pragma scop
S:      set_diagonal(n, A, 0.f);
        for (int i = 0; i < n; ++i)
                for (int j = i + 1; j < n; ++j)
T:                      A[i][j] += A[i][j - 1] + 1;
#pragma endscop
}
```

Instance set: $\{ \mathtt{U}[i] : 0 \leqslant i < n; \mathtt{T}[i,j] : 0 \leqslant i < j < n \}$

Access Relations and Function Calls

```
void set_diagonal(int n,
        float A[const restrict static n][n], float v)
{
        for (int i = 0; i < n; ++i)
U:              A[i][i] = v;
}

void f(int n, float A[const restrict static n][n])
{
#pragma scop
S:      set_diagonal(n, A, 0.f);
        for (int i = 0; i < n; ++i)
                for (int j = i + 1; j < n; ++j)
T:                      A[i][j] += A[i][j - 1] + 1;
#pragma endscop
}
```

May-write: $\{ \texttt{S}[] \to \texttt{A}[i,i] : 0 \leqslant i < n; \texttt{T}[i,j] \to \texttt{A}[i,j] : 0 \leqslant i < j < n \}$
Must-write: $\{ \texttt{S}[] \to \texttt{A}[i,i] : 0 \leqslant i < n; \texttt{T}[i,j] \to \texttt{A}[i,j] : 0 \leqslant i < j < n \}$

Access Relations and Structures

[26]

```
struct s {
        int a;
        int b;
};

int f()
{
        struct s a, b[10];

S:      a.b = 57;
T:      a.a = 42;
        for (int i = 0; i < 10; ++i)
U:              b[i] = a;
}
```

Must-write: $\{ \texttt{S}[] \to \texttt{a\_b}[\texttt{a}[] \to \texttt{b}[]]; \texttt{T}[] \to \texttt{a\_a}[\texttt{a}[] \to \texttt{a}[]];$
$\texttt{U}[i] \to \texttt{b\_a}[\texttt{b}[i] \to \texttt{a}[]]; \texttt{U}[i] \to \texttt{b\_b}[\texttt{b}[i] \to \texttt{b}[]] \}$

Summary Functions

[2, 26]

Analysis of accesses in called function may be inaccurate or even infeasible

- missing body (library function without source)
- unstructured control
- aliasing
- pattern inside dynamic control is ignored
- additional information not explicitly expressed in code

⇒ explicitly specify accesses in summary function pencil

Summary Function Example

```
int f(int i); int maybe(); struct s { int a; };
void set_odd_summary(int n, struct s A[static n]) {
        for (int i = 1; i < n; i += 2)
                if (maybe())
                        A[i].a = 0;
}
__attribute__((pencil_access(set_odd_summary)))
void set_odd(int n, struct s A[static n])
{
        for (int i = 0; i < n; ++i)
                A[2 * f(i) + 1].a = i;
}
void foo(int n, struct s B[static 2 * n])
{
#pragma scop
S:      set_odd(2 * n, B);
#pragma endscop
}
```

May-write: $\{ \texttt{S}[] \to \texttt{B\_a}[\texttt{B}[i] \to \texttt{a}[]] : 0 \leqslant i < 2n \wedge i \bmod 2 = 1 \}$

Context

The context collects constraints on the symbolic constants

- derived by pet
  - exclude values that result in undefined behavior
    - ⋆ negative array sizes
    - ⋆ out-of-bounds accesses
    - ⋆ signed integer overflow
  - `__builtin_assume` or `__pencil_assume` pencil
    - ⇒ any constraint can be specified
    - ⇒ only quasi-affine constraints on symbolic constants are exploited
- specified on PPCG command line
  - `--ctx`
  - `--assume-non-negative-parameters`

Main purpose: simplify generated AST

Dependence analysis in isl

[27, 28]

isl contains generic dependence analysis engine
⇒ determines dependence relations between "sources" and "sinks"

Input:

- Sink $K : I \to D$
- May-source $Y : I \to D$
- Kill $L : I \to D$
- Schedule S on I ⇒ defines "before" and "intermediate"

Output:

- May-dependence relation: triples $(\mathbf{i}, \mathbf{k}, \mathbf{a})$
  - **i** has a may-source to **a**
  - **k** has a sink to **a**
  - **i** is scheduled before **k**
  - there is no intermediate kill to **a**
- May-no-source: sinks **k** → **a** with no kill to **a** before **k**

Dependence analysis in PPCG

[28]

isl:

- May-dependence relation: triples $(\mathbf{i}, \mathbf{k}, \mathbf{a})$
  - **i** has a may-source to **a**
  - **k** has a sink to **a**
  - **i** is scheduled before **k**
  - there is no intermediate kill to **a**
- May-no-source: sinks **k** → **a** with no kill to **a** before **k**

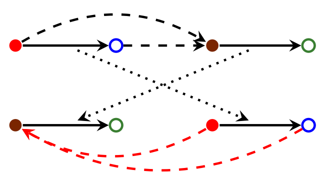
PPCG (without live-range reordering):

- flow dependences (without **a**) and live-in (may-no-source)
  - sink: may-read
  - may-source: may-write
  - kill: must-write
- false dependences (without **a**)
  - sink: may-write
  - may-source: may-read or may-write
  - kill: must-write
- killed writes (without **k**) (⇒ removed from may-write to get live-out)
  - sink: must-write
  - may-source: may-write

Live-Range Reordering

[26, 28]

```
a = f1();
f2(a);
a = f3();
f4(a);
```



⟶: flow
- - →: false

Reordering rejected due to false dependences

Live-range reordering

- allows such live-ranges to be reordered
- using somewhat different classification of dependences
- computed using different calls to the same dependence analysis engine

Pure Kills

[26]

Basic idea:

- Must-writes kill dependences to earlier writes
- Pure kills can also be useful
- Used only as kills during dependence analysis, not as source

Kills can be inserted

- automatically by pet
  - Variable declared within SCoP
    - ⇒ kill at declaration
    - ⇒ kill at end of enclosing block (if within SCoP)
  - Variable declared in scope that contains SCoP, only used inside
    - ⇒ kill at end of SCoP
- manually by the user
  - `__pencil_kill`

pencil

Dependence analysis in PPCG

[28]

`isl`:

- May-dependence relation: triples (**i**, **k**, **a**)
  - **i** has a may-source to **a**
  - **k** has a sink to **a**
  - **i** is scheduled before **k**
  - there is no intermediate kill to **a**
- May-no-source: sinks **k** → **a** with no kill to **a** before **k**

`PPCG` (without live-range reordering):

- flow dependences (without **a**) and live-in (may-no-source)
  - sink: may-read
  - may-source: may-write
  - kill: must-write or pure kill
- false dependences (without **a**)
  - sink: may-write
  - may-source: may-read or may-write
  - kill: must-write
- killed writes (without **k**) (⇒ removed from may-write to get live-out)
  - sink: must-write or pure kill
  - may-source: may-write

Kill Example

```
void f(int n, int A[restrict static n],
        int B[restrict static n])
{
        int t;
#pragma scop
        for (int i = 0; i < n; ++i) {
                t = A[i];
                B[i] = t;
        }
        __pencil_kill(t);
#pragma endscop
}
```

Without kill of `t`, compiler needs to assume `t` may be used after loop

⇒ last write needs to remain last

⇒ limited scheduling freedom (even with live-range reordering)

Note: kill inserted automatically by pet (if `t` not used after SCoP)

Absence of Loop Carried Dependences

[26]

```
void foo(int n, int A[restrict static n][n],
        int B[restrict static n][n])
{
        for (int i = 0; i < n; ++i)
                #pragma pencil independent
                for (int j = 0; j < n; ++j)
                        B[i][A[i][j]] = i + j;
}
```

Assume each row of `A` has distinct elements

⇒ no loop-carried dependences, but `PPCG` cannot tell

⇒ add `#pragma pencil independent`

pencil

Note: not handled very efficiently in current version of `PPCG`

⇒ only add when needed

Optimization Criteria for PPCG

- Two levels of parallelism
  - ⇒ blocks and threads (work groups and work items)
  - ⇒ parallelism

  In PPCG, second level obtained through tiling
  - ⇒ tilability
- Reduced working set for some arrays
  - ⇒ mapping to shared memory or registers

  Obtained through tiling
  - ⇒ tilability
- Reduced data movement
  - ⇒ locality
- Simple schedules
  - ⇒ schedule used in several subsequent steps, including AST generation
  - ⇒ simplicity

Scheduling Constraints

[28]

- Validity $\mathbf{a} \to \mathbf{b}$
  - ⇒ statement instance **b** needs to be executed after **a**
  - ⇒ $f(\mathbf{b}) \geqslant f(\mathbf{a})$
- Proximity $\mathbf{a} \to \mathbf{b}$
  - ⇒ statement instance **b** preferably executed close to **a**
  - ⇒ $f(\mathbf{b}) - f(\mathbf{a})$ as small as possible
- Coincidence $\mathbf{a} \to \mathbf{b}$
  - ⇒ statement instance **b** preferably executed together with **a**
  - ⇒ $f(\mathbf{b}) = f(\mathbf{a})$
  - ⇒ band member only considered "coincident" if it coschedules all pairs
- Conditional validity (live-range reordering)
  - condition $\mathbf{b} \to \mathbf{c}$ (⇜ flow dependences)
  - conditioned validity $\mathbf{a} \to \mathbf{b}$, $\mathbf{c} \to \mathbf{d}$ (⇜ order dependences)

Schedule constraints only relevant if coscheduled by outer nodes
Other schedule constraints are said to be *carried* by some outer node

Dependences and Schedule Constraints

[28]

Traditional dependences

- flow dependences
  - ⇒ validity constraints
  - ⇒ proximity constraints
  - ⇒ coincidence constraints (when parallelism is important)
- false dependences
  - ⇒ validity constraints
  - ⇒ coincidence constraints (when parallelism is important)
  - ⇒ proximity constraints (optional for memory reuse)
- pairs of reads with shared write ("input dependences")
  - ⇒ proximity constraints (optional)

Live-range reordering

- somewhat different classification of dependences
- slightly different mapping to schedule constraints

Current PPCG

- adds false dependences to proximity constraints for historical reasons
- does not consider input dependences
- uses live-range reordering by default

Forced Outer Coincidence Scheduler

Recall:

- Feautrier
  - maximal inner parallelism

    ⇒ carry as many dependences as possible at outer bands
- Pluto
  - tilable bands
  - locality: $f(\mathbf{j}) - f(\mathbf{i})$ small

    ⇒ parallelism as extreme case: $f(\mathbf{j}) - f(\mathbf{i}) = 0$

PPCG uses variant of Pluto-algorithm with Feautrier fallback

⇒ force outer coincidence in each band

⇒ locally fall back to Feautrier if infeasible (single step)

Members in bands constructed by Pluto-algorithm are permutable

⇒ if outer member cannot be coincident, then no member can be

Each step in Feautrier algorithm carries as many dependences as possible

⇒ subsequent application of Pluto more likely to find coincident member

Device Mapping

[31]

Input: schedule tree

If schedule tree contains no coincident band member

⇒ generate pure CPU code

Otherwise:

- select subtree for mapping to the device
  selected subtree is entire schedule tree, except
  - coincidence-free children of outer set node
  - coincidence-free initial children of outer sequence node
- within selected subtree, generate kernels for
  - outermost bands with coincident members
  - maximal coincidence-free subtrees
    ⇒ insert zero-dimensional band node
- add data copying to/from device around selected subtree
- add device initialization and clean-up around entire schedule tree

Data Copying to/from Device

Copy-out:

- take may-writes
- remove writes only needed for dataflow inside selected subtree
- approximate to entire array

May-persist:

- elements that may need to be preserved by selected subtree
- consists of
  - elements that may need to be preserved by entire SCoP
    ⇒ elements not definitely written and not definitely killed
  - elements in potential dataflow across selected subtree

May-not-written: (copy-out $\cap_{\text{ran}}$ may-persist) $\setminus$ must-write

Copy-in: live-in $\cup$ may-not-written

Note: if array elements are structures, then entire structures are copied

Data Copying Example

```
__pencil_kill(A);
for (int i = 0; i < n; i++)
     if (B[i] > 0)
          A[i] = B[i];
```

A may be written

⇒ A in copy-out

A may also *not* be written (completely), but no data can flow across kill

⇒ ~~parts of A may (be expected to) survive~~

⇒ ~~A also needs to be in copy-in~~

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